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A new class of convex games on σ-algebras and the optimal partitioning of measurable spaces

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Abstract We introduce μ-convexity, a new kind of game convexity defined on a σ-algebra of a nonatomic finite measure space. We show that μ-convex games are μ-average monotone. Moreover, we show that μ-average monotone games are totally balanced and their core contains a nonatomic finite signed measure. We apply the results to the problem of partitioning a measurable space among a finite number of individuals. For this problem, we extend some results known for the case of individuals' preferences that are representable by nonatomic probability measures to the more general case of nonadditive representations.

Keywords Nonatomic finite measure · μ-Convex game · μ-Average monotone game · Total balancedness · Fair division · Pareto optimality · α-Fairness · Core stability

1 Introduction

One of the most fundamental solution concepts of cooperative transferable utility (TU) games is the core, defined as the set of feasible payoffs upon which no coalition can improve. It is well known, for instance, that for TU games with a finite number of players, the nonemptiness of the core is equivalent to the balancedness of the game

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(see Bondareva 1963; Shapley 1967) and that the supermodularity (convexity) of the game implies its total balancedness and exactness (see Kelley 1959; Shapley 1971).

One can extend this classical result for games with a finite number of players to TU games with a countably or uncountably infinite number of players. It turns out that sound models of games with an infinite number of players cannot permit the formation of arbitrary coalitions and that considerations of measure theory will play an important role. In the uncountable case, TU games are usually defined on a σ-algebra of subsets of a set of players, and the elements in their core are finitely additive signed measures of bounded variation. Such games have been considered by many studies; see, for example, Aumann and Shapley (1974), Delbaen (1974), Kannai (1969, 1992), Marinacci and Montrucchio (2003, 2004), Schmeidler (1972).

In this paper, we use the notion of the convex combination of measurable sets proposed by Sagara and Vlach (2009, 2010a, b) to introduce a new kind of convexity, μ-convexity, of games defined on a σ-algebra of a nonatomic finite measure space. The convexity of games introduced here inherently differs from the supermodularity of games, but, on the other hand, it is closely related to the average monotone games with a finite number of players proposed by Izqueirdo and Rafels (2001). We show that μ-convex games are μ-average monotone and that μ-average monotone games are totally balanced while their core contains a nonatomic finite signed measure.

We apply these results to the classical problem of partitioning a measurable space among a finite number of individuals along the lines of Dubins and Spanier (1961). We extend the situation where a nonatomic probability measure represents individuals' preferences to the more general situation where they are represented by a nonadditive measure. We show that Pareto optimal α-fair partitions exist if each individual i possesses nonadditive preferences that can be represented by a μ_i-average anti-monotone function.

Moreover, we investigate cooperative games in an exchange economy where the initial individual endowments form a partition of a measurable space. The game studied here is a variant of a market game with a finite dimensional commodity space along the lines of Shapley and Shubik (1969) and allows individuals to form coalitions in the fair division problem. Under the assumption that the utility function of each individual i is μ_i-concave, we show the existence of core stable partitions with nontransferable utility (NTU) and the existence of core stable partitions with transferable utility (TU).

2 A new class of convex games on σ-algebras

2.1 Convex combinations of measurable sets

Let $\mathscr{F}$ be a σ-algebra of subsets of a nonempty set Ω. A measure μ is said to be *nonatomic* if every set $A \in \mathscr{F}$ with $\mu(A) > 0$ includes a set $E \in \mathscr{F}$ such that $0 < \mu(E) < \mu(A)$.

Let $(\Omega, \mathscr{F}, \mu)$ be a nonatomic finite measure space. It follows from the convexity of the range of a nonatomic finite measure that, for every $A \in \mathscr{F}$ and $\alpha \in [0, \mu(A)]$, there exists some $E \subset A$ satisfying $\mu(E) = \alpha$. Further, for every $B \in \mathscr{F}$ and

$\beta \in [\mu(B), \mu(\Omega)]$, there exists some $F \in \mathscr{F}$ with $B \subset F$ such that $\mu(F) = \beta$ (see Halmos 1950, Section 41(2) and (3)).

Let $A \in \mathscr{F}$ and $t \in [0, 1]$ be arbitrarily given, and let $\mathscr{K}_t^{\mu}(A)$ denote the family of measurable subsets of A defined by

$$\mathscr{K}_t^{\mu}(A) = \{E \in \mathscr{F} \mid \mu(E) = t\mu(A),\ E \subset A\}.$$

The nonatomicity of μ implies that $\mathscr{K}_t^{\mu}(A)$ is nonempty for every $A \in \mathscr{F}$ and $t \in [0, 1]$. Note that $E \in \mathscr{K}_t^{\mu}(A)$ if and only if $A \setminus E \in \mathscr{K}_{1-t}^{\mu}(A)$, and that $\mu(A) = 0$ if and only if $\mathscr{K}_t^{\mu}(A)$ contains the empty set for every $t \in [0, 1]$. Denote with $\mathscr{K}_t^{\mu}(A, B)$ the family of sets $C \in \mathscr{F}$ such that C is the union of some disjoint sets $E \in \mathscr{K}_t^{\mu}(A)$ and $F \in \mathscr{K}_{1-t}^{\mu}(B)$.

We shall need the following special case of a more general result on convex combinations of measurable sets for $\mathbb{R}^n$-valued nonatomic finite measures, see Sagara and Vlach (2010b, Theorem 2.1).

Theorem 2.1 *$\mathscr{K}_t^{\mu}(A, B)$ is nonempty for every $A, B \in \mathscr{F}$ and $t \in [0, 1]$.*

2.2 μ-Convex and μ-average monotone games

A real-valued function v on $\mathscr{F}$ satisfying $v(\emptyset) = 0$ is called a *game*; v is *supermodular* (or *convex*) if $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ for every $A, B \in \mathscr{F}$; v is *bounded* if $\sup_{A \in \mathscr{F}} |v(A)| < \infty$.

Definition 2.1 A game v is

(i) *μ-quasiconvex* if $A, B \in \mathscr{F}$ and $t \in [0, 1]$ imply

$$v(C) \leq \max\{v(A), v(B)\} \quad \text{for every } C \in \mathscr{K}_t^{\mu}(A, B)$$

(resp. *μ-quasiconcave if $-v$ is μ-quasiconvex*);

(ii) *μ-convex* if $A, B \in \mathscr{F}$ and $t \in [0, 1]$ imply

$$v(C) \leq tv(A) + (1 - t)v(B) \quad \text{for every } C \in \mathscr{K}_t^{\mu}(A, B)$$

(resp. *μ-concave* if $-v$ is μ-convex).

The notion of μ-(quasi)convexity has recently been introduced by Sagara and Vlach (2009, 2010a, b). The definition bears an obvious resemblance to the definition of (quasi)convex functions on real vector spaces.

Definition 2.2 A game v is *μ-average monotone* if $v(A)\mu(B) \leq v(B)\mu(A)$ for every $A, B \in \mathscr{F}$ with $A \subset B$ (resp. *μ-average anti-monotone* if $-v$ is μ-average monotone).

The notion of average monotonicity was introduced by Izquierdo and Rafels (2001) for games where Ω is a finite set and $\mathscr{F}$ is the algebra of all subsets of Ω. The definition may be written as:

$$\frac{v(A)}{\mu(A)} \leq \frac{v(B)}{\mu(B)} \text{ for every } A, B \in \mathscr{F} \text{ with } A \subset B$$

whenever $\mu(A) > 0$. This states that the average worth of a coalition with respect to the contribution of its players should grow as the coalition increases.

In this paper, we shall be concerned with μ-average monotonicity for nonatomic case, but it is worth noticing that the presented definition is valid for finitely additive measures and requires neither nonatomicity nor countable additivity of μ.

Example 2.1 Let φ be a real-valued function on the range of a nonatomic finite measure μ with $\varphi(0) = 0$. Define the game v_φ on $\mathscr{F}$ by $v_\varphi = \varphi \circ \mu$. Then the following conditions are equivalent: (i) φ is convex; (ii) v_φ is μ-convex. If, moreover, φ is continuous, then the above conditions are equivalent to (iii) v_φ is supermodular.

(i) $\Rightarrow$ (ii): Because $C \in \mathscr{K}_t^\mu(A, B)$ implies $\mu(C) = t\mu(A) + (1-t)\mu(B)$, if φ is convex, then, for every $C \in \mathscr{K}_t^\mu(A, B)$ and $t \in [0, 1]$, we have:

$$\begin{aligned} v_\varphi(C) = \varphi(t\mu(A) + (1-t)\mu(B)) &\leq t\varphi(\mu(A)) + (1-t)\varphi(\mu(B)) \\ &= t v_\varphi(A) + (1-t) v_\varphi(B). \end{aligned}$$

(ii) $\Rightarrow$ (i): Suppose that the function φ is such that the game v_φ is μ-convex. Choose $x, y \in [0, \mu(\Omega)]$ and $t \in [0, 1]$ arbitrarily. By the nonatomicity of μ, there exist A and B in $\mathscr{F}$ such that $\mu(A) = x$ and $\mu(B) = y$. Then, by Theorem 2.1, there exist $E \in \mathscr{K}_t^\mu(A)$ and $F \in \mathscr{K}_{1-t}^\mu(B)$ such that $E \cap F = \emptyset$. We then have:

$$\begin{aligned} \varphi(tx + (1-t)y) = \varphi(t\mu(A) + (1-t)\mu(B)) &= \varphi(\mu(E) + \mu(F)) = v_\varphi(E \cup F) \\ &\leq t v_\varphi(A) + (1-t) v_\varphi(B) = t\varphi(x) + (1-t)\varphi(y). \end{aligned}$$

(iii) $\Leftrightarrow$ (i): Note that the continuous function φ is convex if and only if φ has *increasing differences*; that is, $x, y \in [0, \mu(\Omega)]$, $x \leq y$, $x + z, y + z \in [0, \mu(\Omega)]$ and $z \geq 0$ imply $\varphi(x + z) - \varphi(x) \leq \varphi(y + z) - \varphi(y)$. The supermodularity of v_φ is equivalent to that v_φ has *increasing differences* in the sense that $v_\varphi(A \cup E) - v_\varphi(A) \leq v_\varphi(B \cup E) - v_\varphi(B)$ for every $A, B, E \in \mathscr{F}$ with $A \subset B$ and $B \cap E = \emptyset$ (see, for a proof, Marinacci and Montrucchio 2004, Proposition 4.15). Thus, by using the nonatomicity of μ, it is easy to see that φ has increasing differences if and only if v_φ is supermodular.

2.3 Relationships between classes of games

Denote by $\Gamma_{\mathrm{AM}}^\mu(\Omega, \mathscr{F})$, $\Gamma_{\mathrm{C}}^\mu(\Omega, \mathscr{F})$ and $\Gamma_{\mathrm{QC}}^\mu(\Omega, \mathscr{F})$ respectively the space of μ-average monotone games, the space of μ-convex games and the space of μ-quasiconvex games on a measurable space $(\Omega, \mathscr{F})$.

Theorem 2.2 $\Gamma^{\mu}_{\mathrm{C}}(\Omega, \mathscr{F}) \subsetneq \Gamma^{\mu}_{\mathrm{AM}}(\Omega, \mathscr{F}) \cap \Gamma^{\mu}_{\mathrm{QC}}(\Omega, \mathscr{F})$ *for every nonatomic finite measure* μ *on* $\mathscr{F}$.

Proof To prove the required inclusion, it suffices to show only that μ-convex games are μ-average monotone because μ-convex games are obviously μ-quasiconvex. Let v be a μ-convex game. Choose $A, B \in \mathscr{F}$ arbitrarily such that $A \subset B$. If $\mu(B) = 0$, then $v(A)\mu(B) = v(B)\mu(A) = 0$ is trivially true. If $\mu(B) > 0$, we can define $t = \frac{\mu(A)}{\mu(B)} \in [0, 1]$. Because $v(\emptyset) = 0$ and $A \in \mathscr{K}^{\mu}_t(B, \emptyset)$, we have $v(A) \leq tv(B) + (1-t)v(\emptyset) = tv(B)$ by the μ-convexity of v, which yields $v(A)\mu(B) \leq v(B)\mu(A)$.

To show that the inclusion is proper, consider the function φ defined on the range of μ by

$$\varphi(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2}\mu(\Omega), \\ 2x - \frac{1}{2}\mu(\Omega) & \text{otherwise.} \end{cases}$$

Then φ is average monotone in the sense that $\frac{\varphi(x)}{x}$ is nondecreasing, and quasiconvex. Hence, v_φ is μ-average monotone and μ-quasiconvex. To show that v_φ is not μ-convex, choose $x = \frac{3}{8}\mu(\Omega)$ and $y = \frac{5}{8}\mu(\Omega)$ and $t = \frac{1}{2}$. By the nonatomicity of μ, there exist A, B, C in $\mathscr{F}$ such that $x = \mu(A)$, $y = \mu(B)$, $C \in \mathscr{K}^{\mu}_{1/2}(A, B)$ and $\mu(C) = \frac{1}{2}\mu(\Omega)$. By simple calculation, $v_\varphi(A) = 0$ and $v_\varphi(B) = \frac{3}{4}\mu(\Omega)$. We then have

$$v_\varphi(C) = \frac{1}{2}\mu(\Omega) > \frac{3}{8}\mu(\Omega) = \frac{1}{2}v_\varphi(A) + \frac{1}{2}v_\varphi(B).$$

Therefore, v_φ is not μ-convex. □

Independence of μ-quasiconvexity and μ-average monotonicity The following examples show that $\Gamma^{\mu}_{\mathrm{AM}}(\Omega, \mathscr{F})$ and $\Gamma^{\mu}_{\mathrm{QC}}(\Omega, \mathscr{F})$ are not contained in each other. Hence, μ-average monotonicity and μ-quasiconvexity are independent concepts.

Example 2.2 The game v_φ defined by:

$$\varphi(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{3}\mu(\Omega), \\ \frac{1}{3}\mu(\Omega) & \text{if } \frac{1}{3}\mu(\Omega) \leq x < \frac{2}{3}\mu(\Omega), \\ x - \frac{1}{3}\mu(\Omega) & \text{otherwise} \end{cases}$$

is μ-quasiconvex, but not μ-average monotone.

Example 2.3 Let (P_1, P_2) be a partition of Ω with $\mu(P_1) > 0$ and $\mu(P_2) > 0$. Define the game v by

$$v(A) = \mu(A)\mu(A \cap P_1)\mu(A \cap P_2).$$

Then v is obviously μ-average monotone. Let $A \subset P_2$ and $B \subset P_1$ with $\mu(A) > 0$ and $\mu(B) > 0$. We then have $v(A) = v(B) = 0$ by construction. Choose $t \in (0, 1)$ and $C \in \mathscr{K}^{\mu}_t(A, B)$ arbitrarily. Because $C = E \cup F$ with $E \in \mathscr{K}^{\mu}_t(A)$ and $F \in$

$\mathscr{K}^{\mu}_{1-t}(B)$ by Theorem 2.1, we have $v(C)=\mu(C)\mu(C\cap P_1)\mu(C\cap P_2)=(t\mu(A)+(1-t)\mu(B))t(1-t)\mu(A)\mu(B)>0$. This implies that $\max\{v(A),v(B)\}=0<v(C)$, and hence v cannot be μ-quasiconvex.

It is not difficult to construct examples showing the independence of supermodularity and μ-quasiconvexity, and the independence of supermodularity and μ-average monotonicity. For instance, the game v defined in Example 2.3 is a supermodular game that is not μ-quasiconvex.

2.4 Total balancedness of μ-average monotone games

For a given element $A\in\mathscr{F}$, let $\mathscr{F}_A=\{E\cap A\mid E\in\mathscr{F}\}$ be the sub-σ-algebra of $\mathscr{F}$ restricted to A. A game v_A on $\mathscr{F}_A$ is a *subgame* of v if it is of the form $v_A(E)=v(E)$ for $E\subset A$ with $E\in\mathscr{F}$.

We denote by $ba(\Omega,\mathscr{F})$ the space of finitely additive signed measures on $\mathscr{F}$ with bounded variation. A *feasible payoff* of a game v is an element λ in $ba(\Omega,\mathscr{F})$ satisfying $\lambda(\Omega)=v(\Omega)$. The *core* of a game v is defined by:

$$\mathscr{C}(v)=\{\lambda\in ba(\Omega,\mathscr{F})\mid v\leq\lambda \text{ and } \lambda(\Omega)=v(\Omega)\},$$

that is, the core is the set of feasible payoffs upon which no coalition can improve.

Definition 2.3 A game v is:

(i) *balanced* if $\sum_{i=1}^n\alpha_i v(A_i)\leq v(\Omega)$ for every nonnegative real numbers $\alpha_1,\dots,\alpha_n$ and sets $A_1,\dots,A_n$ in $\mathscr{F}$ satisfying $\sum_{i=1}^n\alpha_i\chi_{A_i}=\chi_\Omega$, where χ_{A_i} is the characteristic function of A_i;

(ii) *totally balanced* if every subgame of v is balanced.

A bounded game has a nonempty core if and only if it is balanced (see Marinacci and Montrucchio 2004; Schmeidler 1972).

Theorem 2.3 (i) *A μ-average monotone game is totally balanced.*

(ii) *For every μ-average monotone game v, the nonatomic finite signed measure μ_v given by $\mu_v=\frac{v(\Omega)}{\mu(\Omega)}\mu$ is in the core of v.*

Proof (i) Let v be a μ-average monotone game. Let $\alpha_1,\dots,\alpha_n$ be nonnegative real numbers and choose $A_1,\dots,A_n$ in $\mathscr{F}$ such that $\sum_{i=1}^n\alpha_i\chi_{A_i}=\chi_\Omega$. As $v(A)\mu(\Omega)\leq v(\Omega)\mu(A)$ for every $A\in\mathscr{F}$, we obtain $\sum_{i=1}^n\alpha_i v(A_i)\leq\frac{v(\Omega)}{\mu(\Omega)}\sum_{i=1}^n\alpha_i\mu(A_i)=v(\Omega)$, and hence v is balanced. Because every subgame of v is obviously μ-average monotone, it is also balanced. Therefore, v is totally balanced.

(ii) Given that μ_v dominates v by the above argument and because $\mu_v(\Omega)=v(\Omega)$, it is sufficient to show that μ_v belongs to $ba(\Omega,\mathscr{F})$. This follows from the fact that the total variation of μ_v is bounded by $2|\mu_v(\Omega)|$ (see Dunford and Schwartz 1958, Lemma III.1.5). □

Let $\Gamma_{\mathrm{TB}}(\Omega,\mathscr{F})$ be the space of totally balanced games on $(\Omega,\mathscr{F})$.

Corollary 2.1 $\Gamma^{\mu}_{\mathrm{C}}(\Omega, \mathscr{F}) \subset \Gamma^{\mu}_{\mathrm{AM}}(\Omega, \mathscr{F}) \subset \Gamma_{\mathrm{TB}}(\Omega, \mathscr{F})$ *for every nonatomic finite measure space* $(\Omega, \mathscr{F}, \mu)$.

One may expect that μ-quasiconvex games have nonempty cores. However, this is not the case as the following example shows.

Example 2.4 Define the game v by:

$$v(A) = \begin{cases} -1 & \text{if } \mu(A) = \mu(\Omega), \\ 0 & \text{otherwise.} \end{cases}$$

Let $A, B \in \mathscr{F}, C \in \mathscr{K}_t^{\mu}(A, B)$ and $t \in [0, 1]$ be arbitrarily given. If $\mu(A) = \mu(B) = \mu(\Omega)$, then $\mu(C) = t\mu(A) + (1 - t)\mu(B) = \mu(\Omega)$, and hence $v(C) = \max\{v(A), v(B)\} = -1$. If $\mu(A) < \mu(\Omega)$ or $\mu(B) < \mu(\Omega)$, then $v(C) \leq 0 = \max\{v(A), v(B)\}$. Therefore, v is μ-quasiconvex.

Suppose that there exists an element $\lambda \in ba(\Omega, \mathscr{F})$ that belongs to the core of v. Take any $A \in \mathscr{F}$ with $0 < \mu(A) < \mu(\Omega)$. Then $\lambda(A) \geq 0$ and $\lambda(\Omega \backslash A) \geq 0$ by the construction of v. On the other hand, $-1 = \lambda(\Omega) = \lambda(A) + \lambda(\Omega \backslash A) \geq 0$, a contradiction.

3 Partitioning of a measurable space

3.1 Pareto optimal α-fair partitions

There are n individuals, each of whom is indexed by $i = 1, \ldots, n$. Each individual's preferences on $\mathscr{F}$ are given by a set function $v_i : \mathscr{F} \to \mathbb{R}$, called a *utility function*, in terms of which the inequality $v_i(A) \geq v_i(B)$ means that individual i prefers A to B or is indifferent between A and B. A set function v_i is *normalized* if $0 \leq v_i \leq 1$, $v_i(\emptyset) = 0$ and $v_i(\Omega) = 1$.

Measures $\mu_1, \ldots, \mu_n$ are *mutually absolutely continuous* if $\mu_j(A) = 0$ for some j with $A \in \mathscr{F}$ implies $\mu_i(A) = 0$ for each $i = 1, \ldots, n$.

A *partition* of Ω is an n-tuple of disjoint sets in $\mathscr{F}$ whose union is Ω. We denote the set of partitions of Ω by $\mathscr{P}^n$.

Let Δ^{n-1} denote the $(n-1)$-dimensional unit simplex in $\mathbb{R}^n$; that is:

$$\Delta^{n-1} = \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i \geq 0,\ i = 1, \ldots, n \right\}.$$

A generic element in Δ^{n-1} is denoted by $\alpha = (\alpha_1, \ldots, \alpha_n)$.

For nonnegative set functions $v_1, \ldots, v_n$, we define the *utility possibility set* V^n_+ by:

$$V^n_+ = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \exists (A_1, \ldots, A_n) \in \mathscr{P}^n : \\ 0 \leq x_i \leq v_i(A_i),\ i = 1, \ldots, n \end{array} \right\}.$$

Definition 3.1 Let $v_1, \ldots, v_n$ be normalized set functions. A partition $(A_1, \ldots, A_n)$ of Ω is:

(i) *α-fair* if $v_i(A_i) \geq \alpha_i$ for each $i = 1, \dots, n$;
(ii) *Pareto optimal* if there exists no partition $(B_1, \dots, B_n)$ of Ω such that $v_i(A_i) \leq v_i(B_i)$ for each $i = 1, \dots, n$ and $v_j(A_j) < v_j(B_j)$ for some j.

The following weaker notion of the strict monotonicity and continuity from below for set functions on $\mathscr{F}$ was proposed by Sagara (2008).

Definition 3.2 A set function $v : \mathscr{F} \to \mathbb{R}$ is:

(i) *monotone* if $v(A) \leq v(B)$ for every $A, B \in \mathscr{F}$ with $A \subset B$;
(ii) *strictly μ-monotone* if $v(A) < v(B)$ for every $A, B \in \mathscr{F}$ with $A \subset B$ and $\mu(A) < \mu(B)$.

Definition 3.3 A set function v is *μ-continuous from below* if $v(A^\nu) \to v(A)$ for every sequence $\{A^\nu\}$ in $\mathscr{F}$ and element $A \in \mathscr{F}$ with $A^1 \subset A^2 \subset \cdots \subset A$ and $\mu(A \setminus \bigcup_{\nu=1}^\infty A^\nu) = 0$.

It is clear that monotonicity, μ-strict monotonicity and μ-continuity from below are automatically satisfied whenever $v = \mu$.

Theorem 3.1 *Let $\mu_1, \dots, \mu_n$ be nonatomic probability measures that are mutually absolutely continuous and let $v_1, \dots, v_n$ be normalized set functions that are monotone, μ_i-continuous from below and μ_i-average anti-monotone for each $i = 1, \dots, n$. If some v_i is strictly μ_i-monotone and V_+^n is closed, then for every $(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$, there exists a Pareto optimal α-fair partition.*

Proof Note that for every $(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$, there exists a partition $(A_1, \dots, A_n)$ of Ω such that $\mu_i(A_i) \geq \alpha_i$ for each i (see Dubins and Spanier 1961). Because $-v_i$ is a μ_i-average monotone game, we have $-v_i \leq (\mu_i)_{-v_i}$ by Theorem 2.3, which yields $v_i \geq \mu_i$ for each i. As a result, solutions to the following maximization problem:

$$\begin{aligned} &\max x_1 \\ &\text{s.t. } x_i \geq \alpha_i, \quad i = 1, \dots, n \\ &(x_1, \dots, x_n) \in V_+^n \end{aligned} \tag{P_α}$$

exist for every $\alpha \in \Delta^{n-1}$ by the compactness of V_+^n. Here, we assume without loss of generality that v_1 is strictly μ_1-monotone.

Take any solution $(x_1, \dots, x_n)$ to the problem (P_α). Then there exists a partition $(A_1, \dots, A_n)$ of Ω such that $v_i(A_i) \geq \alpha_i$ for each i. It suffices to show that $(A_1, \dots, A_n)$ is Pareto optimal. Suppose to the contrary that there exists a partition $(B_1, \dots, B_n)$ of Ω such that $v_i(A_i) \leq v_i(B_i)$ for each i and $v_j(A_j) < v_j(B_j)$ for some j. If $j = 1$, then $x_1 \leq v_1(A_1) < v_1(B_1)$, which obviously violates the fact that $(x_1, \dots, x_n)$ is a solution to (P_α). Thus, we shall investigate the case for $j \neq 1$.

As v_j is μ_j-continuous from below and μ_j is nonatomic, there exists a measurable subset F of B_j such that $v_j(A_j) < v_j(B_j \setminus F)$ and $\mu_j(F) > 0$. By the mutual absolute continuity of each μ_i, we have $\mu_i(F) > 0$ for each i. Define a partition $(C_1, \dots, C_n)$ of Ω by $C_1 = B_1 \cup F$, $C_j = B_j \setminus F$ and $C_i = B_i$ for $i \neq 1, j$. The strict μ_1-monotonicity of v_1 implies that $\alpha_1 \leq x_1 \leq v_1(A_1) \leq v_1(B_1) < v_1(C_1)$, which contradicts the fact that $(x_1, \dots, x_n)$ is a solution to (P_α) in view of $v_i(C_i) \geq \alpha_i$ for each i. □

Some remarks are in order.

To derive Pareto optimality in Theorem 3.1, it suffices to impose strict μ_i-monotonicity at least on one individual's utility function. This requirement is not needed whenever $v_i = \mu_i$ for some i, in which case strict μ_i-monotonicity is automatically satisfied for such individual.

Note that if each v_i is a nonatomic probability measure, then the theorem of Dvoretsky et al. (1951), a variant of Lyapunov's convexity theorem, guarantees the convexity and compactness of V_+^n (see Sagara 2006). In this case, the assumptions in Theorem 3.1 impose nothing on v_i and V_+^n. Therefore, Theorem 3.1 is an extension of the classical result of Dubins and Spanier (1961). In particular, the introduction of μ_i-average anti-monotonicity is novel in the literature.

Dall'Aglio and Maccheroni (2005) and Maccheroni and Marinacci (2003) obtained the existence of α-fair partitions under nonatomic submodular capacities. For a characterization of other solution concepts regarding efficiency and fairness, see Sagara (2008, 2010).

3.2 Core stable partitions with NTU

Denote by $N = \{1, \dots, n\}$ the finite set of individuals. A nonempty subset of N is called a *coalition*. We denote the collection of coalitions by $\mathscr{N}$. Let $(\Omega_1, \dots, \Omega_n)$ be an *initial partition* of Ω in which individual i is endowed with a measurable subset Ω_i of Ω. A partition $(A_1, \dots, A_n)$ of Ω is an *S-partition* if $\bigcup_{i \in S} A_i = \bigcup_{i \in S} \Omega_i$ for coalition S.

As we no longer assume the nonnegative of $v_1, \dots, v_n$ in the sequel, the utility possibility set under consideration becomes:

$$V^n = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \exists (A_1, \dots, A_n) \in \mathscr{P}^n: \\ x_i \le v_i(A_i),\ i = 1, \dots, n \end{array} \right\}.$$

Definition 3.4 A coalition S with NTU *improves upon* a partition $(A_1, \dots, A_n)$ if there exists an S-partition $(B_1, \dots, B_n)$ such that:

$$v_i(A_i) < v_i(B_i) \quad \text{for each } i \in S.$$

A partition that cannot be improved upon by any coalition with NTU is a *core stable partition* with NTU.

The market game $V : \mathscr{N} \to 2^{\mathbb{R}^n}$ with NTU corresponding to the exchange economy is given by:

$$V(S) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \exists S\text{-partition } (A_1, \dots, A_n) : x_i \le v_i(A_i)\ \forall i \in S\}.$$

Lemma 3.1 *Let $\mu_1, \dots, \mu_n$ be nonatomic finite measures. If v_i is μ_i-concave for each $i = 1, \dots, n$, then V^n is convex in $\mathbb{R}^n$.*

Proof Take $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in V^n and $t \in (0, 1)$ arbitrarily. Let $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ be partitions satisfying $x_i \le v_i(A_i)$ and $y_i \le v_i(B_i)$

for each i. By Sagara and Vlach (2010a, Theorem 2.2), there exists a partition $(C_1, \ldots, C_n)$ such that $C_i \in \mathscr{K}_t^{\mu_i}(A_i, B_i)$ for each i. The μ_i-concavity of v_i implies that $tx_i + (1-t)y_i \le tv_i(A_i) + (1-t)v_i(B_i) \le v_i(C_i)$ for each i. Therefore, $tx + (1-t)y \in V^n$. □

Theorem 3.2 *Let $\mu_1, \ldots, \mu_n$ be nonatomic finite measures. If v_i is bounded and μ_i-concave for each $i = 1, \ldots, n$ and V^n is closed, then there exists a core stable partition with NTU.*

Proof Define the n-person game $\tilde{V} : \mathscr{N} \to 2^{\mathbb{R}^n}$ with NTU by:

$$\tilde{V}(S) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \exists (A_1, \ldots, A_n) \in \mathscr{P}^n : x_i \le v_i(A_i) \ \forall i \in S\}.$$

Then $\tilde{V}$ is an extension of V in the sense that $V(S) \subset \tilde{V}(S)$ for every $S \in \mathscr{N}$ and $V(N) = \tilde{V}(N) = V^n$. We show that the core of $\tilde{V}$, denoted by $\mathscr{C}(\tilde{V})$ and defined by:

$$\mathscr{C}(\tilde{V}) = \{(x_1, \ldots, x_n) \in \tilde{V}(N) \mid \nexists S \in \mathscr{N} \ \nexists y \in \tilde{V}(S) : x_i < y_i \ \forall i \in S\},$$

is nonempty.

It is easy to see that each $\tilde{V}(S)$ is *comprehensive from below*, i.e., $x \le y$ and $y \in \tilde{V}(S)$ imply $x \in \tilde{V}(S)$; $x \in \mathbb{R}^n$, $y \in \tilde{V}(S)$ and $x_i = y_i$ for each $i \in S$ imply $x \in \tilde{V}(S)$; each $\tilde{V}(S)$ satisfies the following condition: for each $S \in \mathscr{N}$ there exists a constant M_S such that $x_i \le M_S$ for every $x \in \tilde{V}(S)$ and $i \in S$.

We also show that $\tilde{V}$ is a balanced game. To this end, let $\mathscr{B}$ be a balanced family of $\mathscr{N}$ with balanced weights $\{\lambda^S \ge 0 \mid S \in \mathscr{B}\}$ and let $\mathscr{B}_i = \{S \in \mathscr{B} \mid i \in S\}$. We then have $\sum_{S \in \mathscr{B}_i} \lambda^S = 1$ for each $i = 1, \ldots, n$. Define:

$$\chi_i^S = \begin{cases} 1 & \text{if } S \in \mathscr{B}_i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t^S = \frac{1}{n} \sum_{i \in N} \lambda^S \chi_i^S.$$

We then have:

$$\sum_{S \in \mathscr{B}} t^S = \frac{1}{n} \sum_{S \in \mathscr{B}} \sum_{i \in N} \lambda^S \chi_i^S = \frac{1}{n} \sum_{i \in N} \sum_{S \in \mathscr{B}_i} \lambda^S = 1.$$

Choose any $x = (x_1, \ldots, x_n) \in \bigcap_{S \in \mathscr{B}} \tilde{V}(S)$. Then for every $S \in \mathscr{B}$, there exists a partition $(A_1^S, \ldots, A_n^S)$ such that $x_i \le v_i(A_i^S)$ for each $i \in S$. As $x^S = (v_1(A_1^S), \ldots, v_n(A_n^S))$ belongs to V^n for every $S \in \mathscr{B}$, we have $y = \sum_{S \in \mathscr{B}} t^S x^S \in V^n$ because V^n is convex by Lemma 3.1. Since $x \le y$ by construction, there exists a partition $(A_1, \ldots, A_n)$ such that $x_i \le y_i \le v_i(A_i)$ for each $i = 1, \ldots, n$, which implies $(x_1, \ldots, x_n) \in \tilde{V}(N)$. Therefore, $\bigcap_{S \in \mathscr{B}} \tilde{V}(S) \subset \tilde{V}(N)$, and consequently $\tilde{V}$ is balanced.

We next show that $\tilde{V}(S)$ is closed for every $S \in \mathscr{B}$. To this end, let $\{x^\nu\}$ be a sequence in $\tilde{V}(S)$ converging to $x \in \mathbb{R}^n$. Then there exists a partition $(A_1^\nu, \ldots, A_n^\nu)$ such that $x_i^\nu \le v_i(A_i^\nu)$ for each $i \in S$ and ν. Define the vector $y^\nu = (y_1^\nu, \ldots, y_n^\nu) \in \mathbb{R}^n$ by $y_i^\nu = x_i^\nu$ for $i \in S$ and $y_i^\nu = v_i(A_i^\nu)$ for $i \notin S$. By construction, we have $y^\nu \in V^n$

for each ν. Given $\{v_i(A_i^\nu)\}$ is a bounded sequence in $\mathbb{R}$ for each $i = 1, \ldots, n$, the sequence $\{y^\nu\}$ contains a convergent subsequence with the limit $y \in \mathbb{R}^n$. By the closedness of V^n, we have $y \in V^n$. This implies that there exists a partition $(A_1, \ldots, A_n)$ such that $y_i \le v_i(A_i)$ for each $i = 1, \ldots, n$, and hence $y \in \tilde{V}(S)$. Given $y_i = x_i$ for $i \in S$, we obtain $x \in \tilde{V}(S)$. Therefore, $\tilde{V}(S)$ is closed.

Because the balanced game $\tilde{V}$ obviously satisfies sufficient conditions guaranteeing the nonemptiness of the core of $\tilde{V}$ (see Scarf 1967), we can select an element $(x_1, \ldots, x_n)$ in $\mathscr{C}(\tilde{V})$. Then there exists a partition $(A_1, \ldots, A_n)$ such that $x_i \le v_i(A_i)$ for each $i = 1, \ldots, n$. Suppose that $(A_1, \ldots, A_n)$ is not a core stable partition with NTU. Then there exists an S-partition $(B_1, \ldots, B_n)$ such that $v_i(A_i) < v_i(B_i)$ for each $i \in S$. We then have $(v_1(B_1), \ldots, v_n(B_n)) \in \tilde{V}(S)$ and $x_i < v_i(B_i)$ for each $i \in S$, which contradicts the fact that $(x_1, \ldots, x_n)$ is in $\mathscr{C}(\tilde{V})$ □.

Theorem 3.2 is a partial generalization of the result in Hüsseinov (2008) and Sagara (2006), who imposed the requirement that each v_i is a quasiconcave continuous transformation of a nonatomic finite measure of the form $v_i = \varphi_i \circ \mu_i$. In such a case, V^n is automatically closed and bounded from above. In particular, if φ_i is continuous and concave on the range of μ_i, then v_i is bounded and μ_i-concave (see Example 2.1).

3.3 Core stable partitions with TU

Definition 3.5 A coalition S with TU *improves upon* a partition $(A_1, \ldots, A_n)$ if there exists an S-partition $(B_1, \ldots, B_n)$ such that:

$$\sum_{i \in S} v_i(A_i) < \sum_{i \in S} v_i(B_i).$$

A partition that cannot be improved upon by any coalition with TU is a *core stable partition* with TU.

The market game $w : \mathscr{N} \to \mathbb{R}$ with TU corresponding to the exchange economy is given by:

$$w(S) = \max\{x(S) \mid x \in V(S)\},$$

where $x(S) = \sum_{i \in S} x_i$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Theorem 3.3 *Let $\mu_1, \ldots, \mu_n$ be nonatomic finite measures. If v_i is bounded and μ_i-concave for each $i = 1, \ldots, n$ and V^n is closed, then there exists a core stable partition with TU.*

Proof Define the n-person game $\tilde{w} : \mathscr{N} \to \mathbb{R}$ with TU by

$$\tilde{w}(S) = \max\{x(S) \mid x \in V^n\}.$$

Then $\tilde{w}$ is an extension of w in the sense that $w(S) \le \tilde{w}(S)$ for every $S \in \mathscr{N}$ and $w(N) = \tilde{w}(N)$. As V^n is closed and bounded from above, the maximum is indeed attained for every $S \in \mathscr{B}$. The core of the n-person game $\tilde{w}$, denoted by $\mathscr{C}(\tilde{w})$, is defined by:

$$\mathscr{C}(\tilde{w}) = \{x \in \mathbb{R}^n \mid \tilde{w}(S) \le x(S) \ \forall S \in \mathscr{N} \text{ and } x(N) = \tilde{w}(N)\}.$$

We show that $\tilde{w}$ is totally balanced. Let $\mathscr{B}$ be a balanced family of $\mathscr{N}$ and $\{t^S \mid S \in \mathscr{B}\}$ be the weights defined in the proof of Theorem 3.2. Then for every $S \in \mathscr{B}$, there exists a partition $(A_1^S, \ldots, A_n^S)$ such that $\sum_{i \in S} v_i(A_i^S) = \tilde{w}(S)$. Define $x_i = \sum_{S \in \mathscr{B}} t^S v_i(A_i^S)$ for each $i = 1, \ldots, n$. As $(v_1(A_1^S), \ldots, v_n(A_n^S)) \in V^n$, we have $(x_1, \ldots, x_n) \in V^n$ by Lemma 3.1. We then have:

$$\begin{aligned}\tilde{w}(N) \ge x(N) &= \sum_{i \in N} \sum_{S \in \mathscr{B}} t^S v_i(A_i^S) = \sum_{i \in N} \left(\frac{1}{n} \sum_{S \in \mathscr{B}} \sum_{i \in N} \lambda^S \chi_i^S v_i(A_i^S) \right) \\ &= \sum_{S \in \mathscr{B}} \sum_{i \in N} \lambda^S \chi_i^S v_i(A_i^S) = \sum_{S \in \mathscr{B}} \lambda^S \sum_{i \in S} v_i(A_i^S) = \sum_{S \in \mathscr{B}} \lambda^S \tilde{w}(S).\end{aligned}$$

Therefore, $\tilde{w}$ is balanced. It is evident from this argument that every subgame $\tilde{w}_T$ on 2^T of $\tilde{w}$ of the form $\tilde{w}_T(S) = \tilde{w}(S)$ for $S \subset T$ is balanced. Consequently, $\tilde{w}$ is totally balanced.

By the theorem of Bondareva–Shapley (see Bondareva 1963; Shapley 1967), there exists some $(x_1, \ldots, x_n) \in \mathscr{C}(\tilde{w})$. Let $(A_1, \ldots, A_n)$ be a partition such that $\sum_{i \in N} v_i(A_i) = \tilde{w}(N)$. Suppose that $(A_1, \ldots, A_n)$ is not a core stable partition with TU. For some coalition S and some S-partition $(B_1, \ldots, B_n)$, we have:

$$\sum_{i \in S} v_i(A_i) < \sum_{i \in S} v_i(B_i) \le \tilde{w}(S) \le x(S).$$

Because,

$$\sum_{i \in N \setminus S} v_i(A_i) \le \tilde{w}(N \setminus S) \le x(N \setminus S),$$

summing these inequalities in view of $x(N) = \tilde{w}(N)$ yields

$$x(N) = \sum_{i \in N} v_i(A_i) < x(N),$$

which is a contradiction. □

Theorem 3.3 is an extension of the result in Legut (1986) and Sagara (2006) who assumed that each v_i is a concave continuous transformation of a nonatomic finite measure. Legut (1985) proved the existence of core stable partitions with TU for the countable number of individuals with a nonatomic finite measure.

4 Concluding remarks

We have shown in Theorem 2.2 that μ-convex games are μ-average monotone games and in Theorem 2.3 that the nonatomic finite signed measure μ_v is in the core of a game v if v is μ-convex. Because μ_v is just a possible element in the core, there is likely to exist other elements in the core that are unrelated to the nonatomic finite measure μ. A characterization of the core of a μ-convex game is an interesting issue.

Another open problem is the coincidence of the core with the bargaining set for a μ-convex game. For the case of a finite number of players, Izqueirdo and Rafels (2001) demonstrated that the core coincides with the bargaining set for every average monotone game. For the case of an infinite number of players, the counterpart remains unsolved; it seems to be a difficult problem.

A limitation of our notion of μ-convexity might be seen from the observation that it requires nonatomicity of μ in its definition and is not valid for measures on finite sets, unlike μ-average monotonicity. A suitable definition of μ-convexity on finite sets should be explored to extend the applicability.

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