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Aggregation functions and generalized convexity in fuzzy optimization and decision making

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Abstract In this paper triangular norms and conorms are introduced and suitable definitions and properties are mentioned. Then, aggregation functions and their basic properties are defined. The averaging aggregation operators are defined and some interesting properties are derived. Moreover, we have extended concave and quasiconcave functions introducing t-quasiconcave and upper and lower starshaped functions. The main results concerning aggregation of generalized concave functions are presented and some extremal properties of compromise decisions by adopting aggregation operators are derived and discussed.

1 Introduction

When solving practical optimization problems, we often wish to replace large scale problems by smaller scale problems, or multi-criteria problems by single-criteria problems. However, aggregation problems appear in so many different forms and such a wide range of disciplines and applications that necessarily a general theory of aggregation cannot be very deep. As a consequence the practically relevant parts of aggregation theory are usually domain specific, see Ijiri (1971), Malinvaud (1993).

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In this paper we are interested in aggregation of a finite number of real numbers into a single number and its use in designing new classes of generalized convex functions that may be useful in optimization theory and decision analysis.

In decision making, values to be aggregated are typically preference or satisfaction degrees restricted to the unit interval [0, 1]. Here, we consider a decision problem in X, i.e., the problem to find a "best" decision in the set of feasible decisions X with respect to several criteria functions. We study the "optimal" or "compromise" decision $x^* \in X$ maximizing some aggregation of given criteria. The criteria considered here are functions defined on the set X of feasible decisions with the values in the unit interval [0, 1]. Such functions can be interpreted as membership functions of fuzzy subsets of X, or, utility functions on X. Our approach here is, however, general and can be adopted to a more general class of decision problems, see also Dubois et al. (2000), Fodor and Roubens (1994), Klement et al. (2000).

The set X of feasible decisions is supposed to be a convex subset or a generalized convex subset of the n-dimensional Euclidean space $\mathbf{R}^n$. The main subject of our interest is to derive extremal properties of compromise decisions by adopting aggregation operators, and generalized concave criteria. Hence, we extend the well known results for max-min decisions. Our results will be derived for the n-dimensional Euclidean space $\mathbf{R}^n$ with $n \geq 1$. However, some results can be derived only for $\mathbf{R}^1$, denoted here simply by $\mathbf{R}$.

The paper is structured as follows. In Sect. 2 triangular norms and conorms are introduced and suitable definitions and properties are mentioned. Then, aggregation functions and their basic properties are defined, averaging aggregation operators are defined and some interesting properties are derived. In Sects. 3 and 4 we extend concave and quasiconcave functions introducing t-quasiconcave and starshaped functions. In Sect. 5, the main results concerning aggregation of generalized concave functions are presented and discussed, in Sect. 6, some extremal properties of compromise decisions by adopting aggregation operators are derived.

2 Aggregation functions and operators

Once some values on the unit interval [0, 1] are given, we can aggregate them and obtain a new value belonging again to [0, 1]. This can be done in many different ways according to what is expected from such mappings. They are usually called aggregation functions, and they can be roughly divided into three classes, each possessing very distinct behavior and semantics, see Grabisch et al. (2009).

Functions of the first class, *conjunctive type functions*, combine values as if they were related by a logical "and" operation. In other words, the result of combination is high if all individual values are high. Triangular norms are typical examples of conjunctive type aggregations.

On the other hand, *disjunctive type functions* combine values as an "or" operation, so that the result of aggregation is high if some of the values are high. The most common examples of disjunctive type functions are triangular conorms.

Between conjunctive and disjunctive type functions, there is room for the third class of aggregation functions, which are often called *averaging type functions*. They are usually located between minimum and maximum, which are the bounds of the t-norms and t-conorms. Averaging type functions have the property that low values of some criteria can be compensated by high values of the other criteria functions.

There are of course other operators which do not fit into any of these classes. The process of aggregating several real valued functions defined on a common domain, into a real valued function on the same domain is often realized by combining (for each point of the common domain) the values of the individual functions under consideration into a single representative number, the value of the resulting function. Because of the natural correspondence between real intervals $[a, b]$ and $[0, 1]$, many results for functions whose variables are restricted to a common interval $[a, b]$ can be transformed into results for variables restricted to $[0, 1]$, and vice versa. Consequently, for a large variety of problems, the discussion about aggregation functions is sufficiently general when restricted to functions with variables in $[0, 1]$.

2.1 Triangular norms and conorms

To have a sensible aggregation, the functions performing the process of aggregation should have some reasonable properties; for example, symmetry, increasing monotonicity, strict increasing monotonicity, associativity, continuity, and others, see Klement et al. (2000). Some of these requirements are satisfied by the following functions.

Definition 2.1 A function $T : [0, 1]^2 \to [0, 1]$ that is commutative, associative, nondecreasing in each variable and satisfies the boundary condition $T(a, 1) = a$ for every a in $[0, 1]$, is called the *triangular norm* or *t-norm*.

The most popular triangular norms are the *minimum* t-norm T_M, *product* t-norm T_P, *Lukasiewicz* t-norm T_L, and *drastic product* T_D defined by

$$T_M(a, b) = \min\{a, b\}, \tag{2.1}$$

$$T_P(a, b) = a.b, \tag{2.2}$$

$$T_L(a, b) = \max\{0, a + b - 1\}, \tag{2.3}$$

$$T_D(a, b) = \begin{cases} \min\{a, b\} & \text{if } \max\{a, b\} = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

A class of functions closely related to the class of t-norms is the class of functions from $[0, 1]^2$ into $[0, 1]$ defined as follows.

Definition 2.2 A function $S : [0, 1]^2 \to [0, 1]$ that is commutative, associative, nondecreasing in every variable and satisfies the boundary condition $S(a, 0) = a$ for all $a \in [0, 1]$, is called the *triangular conorm* or *t-conorm*.

The functions S_M, S_P, S_L and S_D defined for $a, b \in [0, 1]$ by

$$S_M(a, b) = \max\{a, b\}, \tag{2.5}$$

$$S_P(a, b) = a + b - a \cdot b, \tag{2.6}$$

$$S_L(a, b) = \min\{1, a + b\}, \tag{2.7}$$

$$S_D(a, b) = \begin{cases} \max\{a, b\} & \text{if } \min\{a, b\} = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{2.8}$$

are typical t-conorms. Often, S_M, S_P, S_L and S_D are called the *maximum, probabilistic sum, bounded sum* and *drastic sum*, respectively.

It can easily be verified that, for every triangular norm T, the function T^* from $[0,1]^2$ into $[0,1]$ defined for all $a, b \in [0,1]$ by

$$T^*(a,b) = 1 - T(1-a, 1-b) \tag{2.9}$$

is a t-conorm. The converse statement is also true. Namely, if S is a t-conorm, then the function $S^* : [0,1]^2 \to [0,1]$ defined for all $a, b \in [0,1]$ by

$$S^*(a,b) = 1 - S(1-a, 1-b) \tag{2.10}$$

is a t-norm. The t-conorm T^* and t-norm S^* are called *dual* to the t-norm T and t-conorm S, respectively. It can easily be verified that

$$T_M^* = S_M, \qquad T_P^* = S_P, \qquad T_L^* = S_L, \qquad T_D^* = S_D.$$

Using the commutativity and associativity of t-norms, one can extend them (and analogously also t-conorms) to more than two arguments by the formula

$$T^n(x_1, x_2, \dots, x_n) = T^2(T^{n-1}(x_1, x_2, \dots, x_{n-1}), x_n), \tag{2.11}$$

where $T^2(x_1, x_2) = T(x_1, x_2)$.

A triangular norm T is called *strictly monotone* if it is strictly increasing in the sense that $T(a,b) < T(a',b)$ whenever $a, a', b \in [0,1]$ and $a < a'$, $b > 0$.

A triangular norm T' *dominates* a triangular norm T if

$$T'(T(a,b), T(c,d)) \geq T(T'(a,c), T'(b,d)) \tag{2.12}$$

for all $a, b, c, d \in [0,1]$.

From (2.11) and (2.12), it follows that a triangular norm T' dominates a triangular norm T if and only if

$$T'(T(x_1, y_1), \dots, T(x_m, y_m)) \geq T(T'(x_1, \dots, x_m), T'(y_1, \dots, y_m)) \tag{2.13}$$

for $m > 1$, and all $x_1, \dots, x_m, y_1, \dots, y_m \in [0,1]$.

2.2 Aggregation functions and operators

There exist other useful functions, often called aggregation functions, that are related to or generalizing t-norms or t-conorms. The following is a specialization to our framework of the definition of aggregation functions from the recent book by Grabisch et al. (2009).

Definition 2.3 Let n be an integer, $n > 1$, an *aggregation function* in $[0,1]^n$ is a function $A : [0,1]^n \to [0,1]$ that is nondecreasing in each variable and fulfills the boundary conditions

$$\inf_{x \in [0,1]^n} A(x) = 0 \quad \text{and} \quad \sup_{x \in [0,1]^n} A(x) = 1. \tag{2.14}$$

Definition 2.4 An aggregation function A in $[0,1]^n$ is called

(a) *conjunctive* if

$$A(x_1, \dots, x_n) \leq T_M(x_1, \dots, x_n), \tag{2.15}$$

(b) *disjunctive* if

$$S_M(x_1, \dots, x_n) \le A(x_1, \dots, x_n), \tag{2.16}$$

(c) *compensatory* if

$$T_M(x_1, \dots, x_n) \le A(x_1, \dots, x_n) \le S_M(x_1, \dots, x_n), \tag{2.17}$$

(d) *commutative* if

$$A(x_{\pi(1)}, \dots, x_{\pi(n)}) = A(x_1, \dots, x_n), \tag{2.18}$$

for every permutation π of $\{1, \dots, n\}$,

(e) *idempotent* if

$$A(x, x, \dots, x) = x, \tag{2.19}$$

for all x.

Notice that (2.14) holds if and only if

$$A(0, \dots, 0) = 0 \quad \text{and} \quad A(1, \dots, 1) = 1. \tag{2.20}$$

Some properties can only be defined when combining different dimensions of aggregating functions. That is why we introduce the concept of aggregation operator. By an aggregation operator we understand a sequence of aggregation functions. More formally:

Definition 2.5 An *aggregation operator* is a sequence $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ of aggregation functions where

$$A_1(x) = x \quad \text{for each } x \in [0, 1].$$

Depending on the field of application, other mathematical properties can be requested from aggregation operators, see e.g. Grabisch et al. (2009).

Definition 2.6 An aggregation operator $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ is called

(a) *commutative*, *idempotent*, *continuous*, *compensatory* or *strictly monotone* if, for each $n \ge 2$, the aggregation function A_n is commutative, idempotent, continuous, compensatory or strictly monotone, respectively;

(b) *strict* if A_n is strictly monotone and continuous for all $n \ge 2$;

(c) *associative* if, for all $m, n \ge 2$ and all tuples $(x_1, x_2, \dots, x_m) \in [0, 1]^m$ and $(y_1, y_2, \dots, y_n) \in [0, 1]^n$, we have

$$\begin{aligned} &A_{m+n}(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) \\ &\quad = A_2(A_m(x_1, x_2, \dots, x_m), A_n(y_1, y_2, \dots, y_n)); \end{aligned}$$

(d) *decomposable* if, for all $m, n \ge 2$ and all tuples $(x_1, \dots, x_m) \in [0, 1]^m$ and $(y_1, \dots, y_n) \in [0, 1]^n$, we have

$$\begin{aligned} &A_{m+n}(x_1, \dots, x_m, y_1, \dots, y_n) \\ &\quad = A_{m+n}(A_m(x_1, \dots, x_m), \dots, A_m(x_1, \dots, x_m), y_1, \dots, y_n) \end{aligned} \tag{2.21}$$

where, in the right side, the term $A_m(x_1, x_2, \dots, x_m)$ occurs m times.

We have seen that the commutativity and associativity make it possible to extend t-norms and t-conorms to n-ary operations, with $n > 2$. Therefore, a sequence $\{T^n\}_{n=1}^{\infty}$ where $T^1(x) = x$ for each $x \in [0, 1]$ defines an aggregation operator, and T^n are its aggregation functions. For the sake of simplicity, when there is no danger of a confusion, we call this aggregation operator also a t-norm and denote it by the original symbol T. In other words, when speaking about a t-norm T or t-conorm S as an aggregation operator, we always have in mind the corresponding sequence $\{T^n\}_{n=1}^{\infty}$ or $\{S^n\}_{n=1}^{\infty}$, respectively. For the same reason, we shall sometimes omit the index n in the aggregation function A_n. Considering this convention in the following propositions, we obtain some characterizations of the previously defined properties.

Every t-norm and every t-conorm is a commutative and associative aggregation operator.

Proposition 2.1 *Let $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator and let $\psi : [0, 1] \to [0, 1]$ be a strictly increasing or strictly decreasing bijection. Then $A^{\psi} = \{A_n^{\psi}\}_{n=1}^{\infty}$ defined by $A_n^{\psi}(x_1, x_2, \dots, x_n) = \psi^{-1}(A_n(\psi(x_1), \dots, \psi(x_n)))$ for all $n = 1, 2, \dots$ and all tuples $(x_1, x_2, \dots, x_n) \in [0, 1]^n$, is an aggregation operator.*

Continuity of aggregation operators play an important role in applications. The following proposition shows that for continuity of commutative aggregation operators it is sufficient that they are continuous in a single variable only. The proof of the following proposition can be found in Ramík and Vlach (2001).

Proposition 2.2 *Let $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ be a commutative aggregation operator. Then $\mathcal{A}$ is continuous if and only if, for each $n = 1, 2, \dots$, the aggregation function A_n is continuous in its first variable x_1.*

2.3 Averaging aggregation operators

Perhaps, even more popular aggregation operators than triangular norms and co-norms are the means: the *arithmetic mean* $M = \{M_n\}_{n=1}^{\infty}$, the *geometric mean* $G = \{G_n\}_{n=1}^{\infty}$, the *harmonic mean* $H = \{H_n\}_{n=1}^{\infty}$ and the *root-power mean* $M^{(\alpha)} = \{M_n^{(\alpha)}\}_{n=1}^{\infty}$, given by

$$M_n(x_1, x_2, \dots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{2.22}$$

$$G_n(x_1, x_2, \dots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \tag{2.23}$$

$$H_n(x_1, x_2, \dots, x_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}, \tag{2.24}$$

$$M_n^{(\alpha)}(x_1, x_2, \dots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right)^{1/\alpha}, \quad \alpha \neq 0. \tag{2.25}$$

All these operators are commutative, idempotent and continuous, none of them is associative. The root-power mean operators $M^{(\alpha)}$, $\alpha > 0$, are strict, whereas G and H are not strict. Notice that $M = M^{(1)}$ and $H = M^{(-1)}$. It can be verified that

$$M_n^{(0)}(x_1, x_2, \dots, x_n) = \lim_{\alpha \to 0} M^{(\alpha)}(x_1, x_2, \dots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n},$$

$$M_n^{(-\infty)}(x_1, x_2, \dots, x_n) = \lim_{\alpha \to -\infty} M^{(\alpha)}(x_1, x_2, \dots, x_n)$$
$$= \min\{x_i \mid i = 1, \dots, n\},$$
$$M_n^{(+\infty)}(x_1, x_2, \dots, x_n) = \lim_{\alpha \to +\infty} M^{(\alpha)}(x_1, x_2, \dots, x_n)$$
$$= \max\{x_i \mid i = 1, \dots, n\}.$$

The next proposition says that the operators (2.22)–(2.25) are all compensatory. It says even more, namely, that the class of idempotent aggregation operators is exactly the same as the class of compensatory operators. The proof of this result is elementary and can be found in Fodor and Roubens (1994).

Proposition 2.3 *An aggregation operator is idempotent if and only if it is compensatory.*

The following proposition clarifies the relationships between some other properties introduced in Definition 2.6. The proof can be found also in Fodor and Roubens (1994).

Proposition 2.4 *Let $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ be a continuous and commutative aggregation operator. Then $\mathcal{A}$ is compensatory, strict and decomposable, if and only if for all $x_1, x_2, \dots, x_n \in [0, 1]$*

$$A_n(x_1, x_2, \dots, x_n) = \psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \psi(x_i)\right), \tag{2.26}$$

with a continuous strictly monotone function $\psi : [0, 1] \to [0, 1]$.

The aggregation operator (2.26) is called the *generalized mean*, or *quasi-arithmetic mean.* It covers a wide range of popular means including those of (2.22)–(2.25). The minimum T_M and the maximum S_M are the only associative and decomposable compensatory aggregation operators.

3 Quasiconcave and starshaped functions

In this section and the following sections we shall deal with our main problem, that is, the aggregation of generalized quasiconcave functions. First, we will look for sufficient conditions that secure some properties of quasiconcavity. For a more detailed treatment of concavity and some of its generalizations, see Avriel et al. (1988) or Ramík and Vlach (2001).

The concepts of concavity, convexity, quasiconcavity, quasiconvexity and quasimonotonicity of a function $f : \mathbf{R}^n \to \mathbf{R}$ can be introduced in several ways. The following definitions are suitable for our purpose. From now on, the set X is supposed to be a nonempty subset of $\mathbf{R}^n$, we shall denote it $X \subseteq \mathbf{R}^n$.

Definition 3.1 Let $X \subseteq \mathbf{R}^n$, a function $f : X \to \mathbf{R}$ is called

(a) *concave on X (CA)* if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \tag{3.1}$$

for every $x, y \in X$ and every $\lambda \in (0, 1)$ with $\lambda x + (1-\lambda)y \in X$;

(b) *strictly concave on* X if

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y) \tag{3.2}$$

for every $x, y \in X$, $x \neq y$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$;

(c) *semistrictly concave on* X if f is concave on X and (3.2) holds for every $x, y \in X$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$ such that $f(x) \neq f(y)$.

A function $f : X \to \mathbf{R}$ is called *convex, strictly convex on* X*, semi-strictly convex on* X if $-f$ is concave on X, strictly concave on X, semistrictly concave on X, respectively.

Definition 3.2 Let $X \subseteq \mathbf{R}^n$, a function $f : X \to \mathbf{R}$ is called

(a) *quasiconcave on* X *(QCA)* if

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$$

for every $x, y \in X$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$ (resp. *quasiconvex on* X if $-f$ is quasiconcave on X);

(b) *strictly quasiconcave on* X if

$$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\} \tag{3.3}$$

for every $x, y \in X$, $x \neq y$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$ (resp. *strictly quasiconvex on* X if $-f$ is strictly quasiconcave on X);

(c) *semistrictly quasiconcave on* X if f is quasiconcave on X and (3.3) holds for every $x, y \in X$ and every $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in X$ such that $f(x) \neq f(y)$.

A function $f : X \to \mathbf{R}$ is called *quasiconvex on* X*, strictly quasiconvex on* X*, semistrictly quasiconvex on* X if $-f$ is quasiconcave on X, strictly quasiconcave on X, semistrictly quasiconcave on X, respectively.

Notice that in Definitions 3.1 and 3.2 the set X is not required to be convex. If in the above definitions the set X is convex, then we obtain the usual definition of (strictly) quasiconcave and (strictly) quasiconvex functions. Observe that if a function is (strictly) concave and (strictly) convex on X, then it is (strictly) quasiconcave and (strictly) quasiconvex on X, respectively, but not vice-versa.

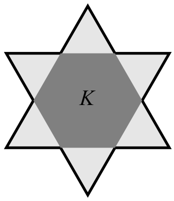
In Definitions 3.1 and 3.2 we introduced the concepts of semistrictly CA functions and semistrictly QCA functions, respectively. The former (the latter) is stronger than the concept of a CA function (QCA function), and weaker than the concept of a strictly CA function (strictly QCA function).

Let X be a convex subset of $\mathbf{R}^n$ and let f be a real valued function on X. By the *upper-level set of* f at a real number α, we understand the set $U(f,\alpha)$ defined by $U(f,\alpha) = \{x \in X : f(x) \geq \alpha\}$. Analogously, the set $L(f,\alpha) = \{x \in X : f(x) \leq \alpha\}$ will be called the *lower-level set of* f at α.

Further on, we shall investigate some properties of local and global extrema of quasiconcave functions.

Proposition 3.1 *Let* $X \subseteq \mathbf{R}^n$ *be a convex set. Let* $f : X \to \mathbf{R}$ *be semistrictly quasiconcave on* X*. If* $x^* \in X$ *is a local maximizer of* f *over* X*, then* x^* *is a global maximizer of* f *over* X*.*

Fig. 1 Starshaped set



Proof Set $\alpha = f(\bar{x})$. Then the upper-level set $U(f, \alpha)$ is convex. Since $\bar{x} \in X$ is a local maximizer, there exists an open ball B with the center at $\bar{x} \in X$, such that $f(x) \le f(\bar{x})$ for all $x \in X \cap B$.

Suppose on contrary that $\bar{x} \in X$ is not a global maximizer. Then there exists $v \in X$, $\bar{x} \neq v$, such that $f(\bar{x}) < f(v)$. It follows that $v \in U(f, \alpha)$ and consequently there exists a segment $\mathbf{S}(\bar{x}, v) = \lambda x + (1 - \lambda)v$ connecting $\bar{x}$ and v, such that $\mathbf{S}(\bar{x}, v) \subset U(f, \alpha)$, set $z = \lambda' x + (1 - \lambda')v \in X \cap B$ for some sufficiently small $\lambda' \in (0, 1)$. As f is semistrictly quasiconcave on X, $f(\lambda' x + (1 - \lambda')v) = f(z) > f(\bar{x})$. Since $z \in X \cap B$, we have $f(z) \le f(\bar{x})$, a contradiction. □

It is well known that a function is quasiconcave on X if and only if all its upper-level sets are convex sets. The following generalization of concave functions is based on this observation generalizing the concept of convex (upper-level) set by introducing starshaped sets.

Definition 3.3 Let $X \subseteq \mathbf{R}^n$ and let y be a point in X. The set X is *starshaped from* y if, for every $x \in X$, the convex hull of the set $\{x, y\}$ is included in X. The set of all points $y \in X$ such that X is starshaped from y is called the *kernel* of X and it is denoted by $\mathrm{Ker}(X)$. The set X is said to be a *starshaped set* if $\mathrm{Ker}(X)$ is nonempty, or if X is empty.

Clearly, X is starshaped if there is a point $y \in X$ such that X is starshaped from y. From the geometric viewpoint, if there exists a point y in X such that for every other point x from X the whole linear segment connecting the points x and y belongs to X, then X is starshaped, see Fig. 1. Evidently, every convex set is starshaped. For a convex set X, we have $\mathrm{Ker}(X) = X$. Moreover, in the 1-dimensional space $\mathbf{R}$, convex sets and starshaped sets coincide.

To introduce starshaped functions, we begin with the following, well known, characterization of quasiconcave and quasiconvex functions by upper-level sets and lower-level sets.

Proposition 3.2 *Let $X \subseteq \mathbf{R}^n$ be a convex set. A function $f : X \to \mathbf{R}$ is quasiconcave on X if and only if all its upper-level sets are convex subsets of $\mathbf{R}^n$. Likewise, f is quasiconvex on X if and only if all its lower-level sets are convex subsets of $\mathbf{R}^n$.*

Proposition 3.2 suggest a way of generalization of quasiconcave and quasiconvex functions. Replacing the requirement of convexity of upper-level sets and lower-level sets in Proposition 3.2 by the requirement of starshapedness, we obtain the following generalization of quasiconcave and quasiconvex functions.

Definition 3.4 Let X be a starshaped subset of $\mathbf{R}^n$. A function $f : X \to \mathbf{R}$ is called

(a) *upper-starshaped on* X if its upper-level sets $U(f, \alpha)$ are starshaped subsets of $\mathbf{R}^n$ for all $\alpha \in \mathbf{R}$;

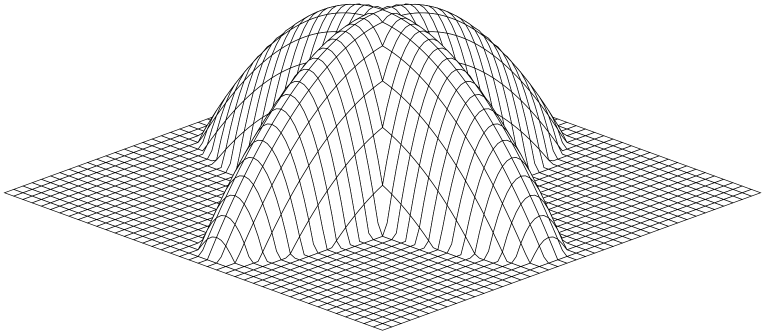


Fig. 2 Starshaped function which is not quasiconcave

(b) *lower-starshaped on* X if its lower-level sets $L(f,\alpha)$ are starshaped subsets of $\mathbf{R}^n$ for all $\alpha \in \mathbf{R}$;
(c) *monotone-starshaped on* X if it is both lower-starshaped and upper-starshaped on X.

It is obvious that if a function $f : X \to \mathbf{R}^n$ is upper-starshaped on X, then the function $-f$ is lower-starshaped on X, and vice-versa. From the fact that each convex set is starshaped it follows that each quasiconcave (quasiconvex) function is upper-starshaped (lower-starshaped), but not vice versa, see Fig. 2. Moreover, each quasimonotone function is monotone-starshaped. Evidently, the classes of quasiconcave (quasiconvex) functions and upper-starshaped (lower-starshaped) functions coincide on $\mathbf{R}^n, n = 1$, as convex sets and starshaped sets on $\mathbf{R}$ coincide. For $n > 1$, this is, however, not true.

Further on, we shall investigate some properties of local and global maxima of upper-starshaped functions on X.

Proposition 3.3 *Let $X \subseteq \mathbf{R}^n$ be a starshaped set. Let $f : X \to \mathbf{R}$ be upper-starshaped on X. If $x^* \in X$ is a strict local maximizer of f over X, then x^* is a strict global maximizer of f over X.*

Proof Set $\alpha = f(\bar{x})$. Then the upper-level set $U(f,\alpha)$ is upper-starshaped. Since $\bar{x} \in X$ is a strict local maximizer, there exists an open ball B with the center at $\bar{x} \in X$, such that $f(x) < f(\bar{x})$ for all $x \in X \cap B$.

Suppose on contrary that $\bar{x} \in X$ is not a strict global maximizer. Then there exists $v \in X$, $\bar{x} \neq v$, such that $f(\bar{x}) < f(v)$. It follows that $v \in U(f,\alpha)$ and consequently there exists a "broken" segment $\mathbf{S}(\bar{x}, v, \lambda)$ connecting $\bar{x}$ and v, such that $\mathbf{S}(\bar{x}, v, \lambda) \subset U(f,\alpha)$. Then $z = \mathbf{S}(\bar{x}, v, \lambda') \in X \cap B$ for some sufficiently small $\lambda' \in (0, 1)$ and $f(\mathbf{S}(\bar{x}, v, \lambda')) = f(z) \geq f(\bar{x})$. Since $z \in X \cap B$, we have $f(z) \leq f(\bar{x})$, a contradiction. □

4 *T*-quasiconcave functions

In contrast to the previous section, we now restrict our attention to functions on $\mathbf{R}^n$ whose range is included in the unit interval [0, 1] of real numbers. Such functions can be interpreted as membership functions of fuzzy subsets of $\mathbf{R}^n$, important in fuzzy optimization and decision making. In this context they are called *fuzzy criteria*. We therefore use several terms and some notation of fuzzy set theory. However, it should be pointed out that such functions arise in more contexts.

In what follows, the Greek letter μ, sometimes with an index, denotes a function that maps $\mathbf{R}^n$ into the interval $[0,1]$. The *core* of μ, denoted by $\mathrm{Core}(\mu)$ is the set of those points in $\mathbf{R}^n$ at which $\mu(x)=1$. If the Core of μ is nonempty, then μ is said to be *upper-normalized*; if the core of $1-\mu$ is nonempty, then μ is said to be *lower-normalized*; and if μ is both upper-normalized and lower-normalized, then it is called *normalized*.

We have introduced quasiconcave (semi)strictly quasiconcave, quasiconvex and (semi) strictly quasiconvex functions in Definition 3.1. First, we extend these notions by using triangular norms and conorms.

Definition 4.1 Let $X \subseteq \mathbf{R}^n$ be a convex set, T be a triangular norm. A function $\mu : \mathbf{R}^n \to [0,1]$ is called

(a) *T-quasiconcave* on X if

$$\mu(\lambda x + (1-\lambda y)) \geq T(\mu(x), \mu(y)) \tag{4.1}$$

for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$;

(b) *strictly T-quasiconcave* on X if

$$\mu(\lambda x + (1-\lambda)y) > T(\mu(x), \mu(y)) \tag{4.2}$$

for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$;

(c) *semistrictly T-quasiconcave* on X if (4.1) holds for every $x, y \in X$, $x \neq y$ and $\lambda \in (0,1)$ and (4.2) holds for every $x, y \in X$ and $\lambda \in (0,1)$ such that $\mu(x) \neq \mu(y)$;

Similarly, we can define (*strictly, semistrictly*) *S-quasiconvex* on X, (*strictly, semistrictly*) *(T,S)-quasimonotone* on X.

Obviously, the class of quasiconcave functions that map $\mathbf{R}^n$ into $[0,1]$ according to Definition 3.2 is exactly the class of T_M-quasiconcave functions according to Definition 4.1. Moreover, since the minimum triangular norm T_M is the maximal t-norm, and the drastic product T_D is the minimal t-norm, we have the following consequence of Definition 4.1.

Proposition 4.1 *Let $X \subseteq \mathbf{R}^n$ be a convex set, μ be a function from $\mathbf{R}^n$ into $[0,1]$, and T be a triangular norm.*

(a) *If μ is (strictly, semistrictly) quasiconcave function on X, then μ is (strictly, semistrictly) T-quasiconcave on X, respectively.*

(b) *If μ is (strictly, semistrictly) T-quasiconcave function on X, then μ is also (strictly, semistrictly) T_D-quasiconcave on X.*

It is easy to show the following relationship between T-quasiconcave and S-quasiconvex functions.

Proposition 4.2 *Let $X \subseteq \mathbf{R}^n$ be a convex set, let T be a triangular norm and let $\mu : \mathbf{R}^n \to [0,1]$ be (strictly, semistrictly) T-quasiconcave on X. Then $\mu^* = 1-\mu$ is (strictly, semistrictly) T^*-quasiconvex on X, where T^* is the t-conorm dual to T.*

Proof Clearly, the claim follows directly from Definition 4.1 and the relation $T^*(a,b) = 1 - T(1-a, 1-b)$. □

It is also easy to show that there exist T-quasiconcave functions that are not quasiconcave (see Ramík and Vlach 2001), and there exist strictly or semistrictly T-quasiconcave functions that are not strictly or semistrictly quasiconcave. Nevertheless, in the one-dimensional Euclidean space $\mathbf{R}$, the following proposition is of some interest.

Proposition 4.3 *Let $X \subseteq \mathbf{R}$ be a convex set, let T be a triangular norm, and let $\mu : \mathbf{R} \to [0, 1]$ be such that $\mu(\bar{x}) = 1$ for some $\bar{x} \in X$. If μ is (strictly, semistrictly) T-quasiconcave on X, then μ is (strictly, semistrictly) quasiconcave on X.*

Proof We prove only the part concerning T-quasiconcavity. The part concerning strict (semistrict) T-quasiconcavity can be verified analogously. Let λ be in $[0, 1]$ and let x and y in X. Without loss of generality we assume that $x \le y$. Thus $x \le \lambda x + (1 - \lambda)y$. If $\lambda x + (1 - \lambda)y \le \bar{x}$, then there exists $\alpha \in [0, 1]$ such that

$$\lambda x + (1 - \lambda)y = \alpha x + (1 - \alpha)\bar{x}.$$

By T-quasiconcavity of μ on X and properties of triangular norms, we have

$$\mu(\alpha x + (1 - \alpha)\bar{x}) \ge T(\mu(x), \mu(\bar{x})) = T(\mu(x), 1) = \mu(x).$$

Since $\mu(\lambda x + (1 - \lambda)y) = \mu(\alpha x + (1 - \alpha)\bar{x})$ and $\mu(x) \ge \min\{\mu(x), \mu(y)\}$, we have

$$\mu(\lambda x + (1 - \lambda)y) \ge \min\{\mu(x), \mu(y)\}.$$

It remains to show that this inequality holds also in the case that $\bar{x} < \lambda x + (1 - \lambda)y$. Since $\lambda x + (1 - \lambda)y \le y$, we can use an analogous argument. There exists $\beta \in [0, 1]$ such that $\lambda x + (1 - \lambda)y = \beta\bar{x} + (1 - \beta)y$ and consequently

$$\begin{aligned} \mu(\lambda x + (1 - \lambda)y) &= \mu(\beta\bar{x} + (1 - \beta)y) \ge T(\mu(\bar{x}), \mu(y)) \\ &= T(1, \mu(y)) = T(\mu(y), 1) = \mu(y) \\ &\ge \min\{\mu(x), \mu(y)\}. \end{aligned}$$

□

Analogous propositions are valid for S-quasiconvex functions and for (T, S)-quasimonotone functions.

5 Aggregation of fuzzy criteria

In what follows we shall investigate the properties of aggregations of fuzzy criteria given by membership functions of fuzzy sets in $\mathbf{R}^n$ by the help of a t-norm or aggregation function. Particularly, we deal with some sufficient conditions such that the aggregations of fuzzy criteria will become T-quasiconcave or upper-starshaped. The generalized concavity properties are very important in optimization or decision making as they secure that any local maximizer is also a global one. That property will be investigated in the next section.

Proposition 5.1 *Let $X \subseteq \mathbf{R}^n$ be a convex set, let T and T' be t-norms and let $\mu_i : \mathbf{R}^n \to [0, 1]$, $i = 1, 2, \ldots, m$, be T-quasiconcave on X. If T' dominates T, then $A_m : \mathbf{R}^n \to [0, 1]$ defined by $A_m(x) = T'(\mu_1(x), \ldots, \mu_m(x))$ is T-quasiconcave on X.*

Proof As μ_i, $i = 1, 2, \ldots, m$ are T-quasiconcave on X, we have $\mu_i(\lambda x + (1-\lambda)y) \geq T(\mu_i(x), \mu_i(y))$ for every $\lambda \in [0, 1]$ and $x, y \in X$. By monotonicity of T', we obtain

$$A_m(\lambda x + (1-\lambda)y) = T'(\mu_1(\lambda x + (1-\lambda)y), \ldots, \mu_m(\lambda x + (1-\lambda)y)) \geq T'(T(\mu_1(x), \mu_1(y)), \ldots, T(\mu_m(x), \mu_m(y))). \quad (5.1)$$

Using the fact that T' dominates T, we obtain

$$T'(T(\mu_1(x), \mu_1(y)), \ldots, T(\mu_m(x), \mu_m(y))) \geq T(T'(\mu_1(x), \ldots, \mu_m(x)), T'(\mu_1(y), \ldots, \mu_m(y))) = T(\varphi(x), \varphi(y)). \quad (5.2)$$

Combining (5.1) and (5.2), we obtain the required result. □

Corollary 5.1 *Let $X \subseteq \mathbf{R}^n$ be a convex set, let T be a t-norm, and let $\mu_i : \mathbf{R}^n \to [0, 1]$, $i = 1, 2, \ldots, m$, be T-quasiconcave on X. Then $A_j : \mathbf{R}^n \to [0, 1]$, $j = 1, 2$, defined by $A_1(x) = T(\mu_1(x), \ldots, \mu_m(x))$ and $A_2(x) = T_M(\mu_1(x), \ldots, \mu_m(x))$, are also T-quasiconcave on X.*

Proof The proof follows from the preceding proposition and the evident fact that T dominates T and T_M dominates every t-norm T. □

The following results are also of some interest, for proofs, see Ramík and Vlach (2001). Notice that any quasiconcave function on X is T_D-quasiconcave on X and also upper-starshaped on X.

Proposition 5.2 *Let $X \subseteq \mathbf{R}^n$ be a convex set, let T be a t-norm and let $\mu_i : \mathbf{R}^n \to [0, 1]$, $i = 1, 2, \ldots, m$, be T-quasiconcave on X such that*

$$\mathrm{Core}(\mu_1) \cap \cdots \cap \mathrm{Core}(\mu_m) \neq \emptyset.$$

Let $A_m : [0, 1]^m \to [0, 1]$ be an aggregation function. Then $\psi : \mathbf{R}^n \to [0, 1]$ defined by $\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x))$ is upper-starshaped on X.

Proposition 5.3 *Let $X \subseteq \mathbf{R}^n$ be a convex set, let T be a t-norm and let $\mu_i : \mathbf{R}^n \to [0, 1]$, $i = 1, 2, \ldots, m$, be T-quasiconcave on X such that*

$$\mathrm{Core}(\mu_1) = \cdots = \mathrm{Core}(\mu_m) \neq \emptyset.$$

Let $A_m : [0, 1]^m \to [0, 1]$ be a strictly monotone aggregation function. Then $\psi : \mathbf{R}^n \to [0, 1]$ defined for $x \in \mathbf{R}^n$ by $\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x))$ is T_D-quasiconcave on X.

The above proposition allows for constructing new T_D-quasiconcave function on $X \subseteq \mathbf{R}^n$ from the original T-quasiconcave functions on $X \subseteq \mathbf{R}^n$ by using a strictly monotone aggregation operator; for example, the t-conorm S_M. It is of interest to note that the condition $\mathrm{Core}(\mu_1) = \cdots = \mathrm{Core}(\mu_m) \neq \emptyset$ is essential for T_D-quasiconcavity of ψ in Proposition 5.3.

The following definition extends the concept of domination between two triangular norms (2.13) to aggregation operators.

Definition 5.1 An aggregation operator $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ *dominates* an aggregation operator $\mathcal{A}' = \{A'_n\}_{n=1}^{\infty}$, if, for all $m \geq 2$ and all tuples $(x_1, \dots, x_m) \in [0, 1]^m$ and $(y_1, \dots, y_m) \in [0, 1]^m$, the following inequality holds

$$A_m(A'_2(x_1, y_1), \dots, A'_2(x_m, y_m)) \geq A'_2(A_m(x_1, x_2, \dots, x_m), A_m(y_1, y_2, \dots, y_m)).$$

The following proposition generalizes Proposition 5.1.

Proposition 5.4 *Let $X \subseteq \mathbf{R}^n$ be a convex set, let $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator, T be a t-norm and let $\mu_i : \mathbf{R}^n \to [0, 1]$, $i = 1, 2, \dots, m$, be T-quasiconcave on X, and let $\mathcal{A}$ dominates T. Then $\varphi : \mathbf{R}^n \to [0, 1]$ defined by*

$$\varphi(x) = A_m(\mu_1(x), \dots, \mu_m(x))$$

is T-quasiconcave on X.

Proof As μ_i , $i = 1, 2, \dots, m$, are T-quasiconcave on X, we have $\mu_i(\lambda x + (1 - \lambda)y) \geq T(\mu_i(x), \mu_i(y))$ for every $\lambda \in (0, 1)$ and each $x, y \in X$. By monotonicity of aggregating mapping A_m, we obtain

$$\begin{aligned} &\varphi(\lambda x + (1 - \lambda)y) \\ &\quad = A_m(\mu_1(\lambda x + (1 - \lambda)y), \dots, \mu_m(\lambda x + (1 - \lambda)y)) \\ &\quad \geq A_m(T(\mu_1(x), \mu_1(y)), \dots, T(\mu_m(x), \mu_m(y))). \end{aligned} \tag{5.3}$$

Using the fact that $\mathcal{A}$ dominates T, we obtain

$$\begin{aligned} &A_m(T(\mu_1(x), \mu_1(y)), \dots, T(\mu_m(x), \mu_m(y))) \\ &\quad \geq T(A_m(\mu_1(x), \dots, \mu_m(x)), A_m(\mu_1(y), \dots, \mu_m(y))) \\ &\quad = T(\varphi(x), \varphi(y)), \end{aligned} \tag{5.4}$$

where $T = T^{(2)}$. Combining (5.3) and (5.4), we obtain the required result. □

6 Extremal properties

In this last section we derive several useful results concerning relations between local and global maximizers of some aggregations of fuzzy criteria. From the point of view of fuzzy multi-criteria decision making, such global maximizers are considered as "compromise" decisions. For this purpose we apply Propositions 3.1 and 3.3 in some combinations with the results on aggregation operators from Propositions 5.2 and 5.4. Some relations between "compromise" decision and Pareto-optimal decision can be found in Ramík and Vlach (2002).

Proposition 6.1 *Let $X \subseteq \mathbf{R}^n$ be a convex set, let T be a t-norm and let $\mu_i : \mathbf{R}^n \to [0, 1]$, $i = 1, 2, \dots, m$, be T-quasiconcave on X such that*

$$\text{Core}\,(\mu_1) \cap \dots \cap \text{Core}\,(\mu_m) \neq \emptyset.$$

Let $A : [0,1]^m \to [0,1]$ *be an aggregation function. If* $\psi : \mathbf{R}^n \to [0,1]$ *defined by* $\psi(x) = A(\mu_1(x), \ldots, \mu_m(x))$ *attains its strict local maximum at* $x^* \in X$, *then* x^* *is a strict global maximizer of* ψ *over* X.

Proof By Proposition 5.2, ψ is upper-starshaped on X. Now, by Proposition 3.3, we obtain the required result. □

Proposition 6.2 *Let* $X \subseteq \mathbf{R}^n$ *be a convex set, let* T *be a t-norm and let* $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1, 2, \ldots, m$, *be* T*-quasiconcave on* X. *Let* $\mathcal{A} = \{A_k\}_{k=1}^{\infty}$ *be an aggregation operator and let* $\mathcal{A}$ *dominate* T. *If* $\psi : \mathbf{R}^n \to [0,1]$ *defined by* $\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x))$ *attains its strict local maximum at* $x^* \in X$, *then* x^* *is a strict global maximizer of* ψ *over* X.

Proof By Proposition 5.4, function ψ is T-quasiconcave on X. Since each T-quasiconcave function on X is upper-starshaped on X, the statement follows from Proposition 3.3. □

Proposition 6.3 *Let* $X \subseteq \mathbf{R}^n$ *be a convex set, let* T *be a t-norm and let* $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1, 2, \ldots, m$, *be semistrictly* T*-quasiconcave on* X. *Let* $\mathcal{A} = \{A_k\}_{k=1}^{\infty}$ *be a strictly monotone aggregation operator and let* $\mathcal{A}$ *dominate* T. *If* $\psi : \mathbf{R}^n \to [0,1]$ *defined by* $\psi(x) = A_m(\mu_1(x), \ldots, \mu_m(x))$ *attains its local maximum at* $x^* \in X$, *then* x^* *is a global maximizer of* ψ *over* X.

Proof By Proposition 5.4, and strict monotonicity of A, function ψ is semistrictly T-quasiconcave on X. Then the statement follows from Proposition 3.1. □

Since each t-norm T dominates T and the minimum t-norm T_M dominates any other t-norm T, we obtain the following results.

Corollary 6.1 *Let* $X \subseteq \mathbf{R}^n$ *be a convex set, let* T *be a t-norm, and let* $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1, 2, \ldots, m$, *be* T*-quasiconcave on* X. *If* $\varphi : \mathbf{R}^n \to [0,1]$ *defined by* $\varphi(x) = T(\mu_1(x), \ldots, \mu_m(x))$ *attains its strict local maximum at* $x^* \in X$, *then* x^* *is a strict global maximizer of* φ *over* X.

Corollary 6.2 *Let* $X \subseteq \mathbf{R}^n$ *be a convex set, let* T *be a strict t-norm, and let* $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1, 2, \ldots, m$, *be semistrictly* T*-quasiconcave on* X. *If* $\varphi : \mathbf{R}^n \to [0,1]$ *defined by* $\varphi(x) = T(\mu_1(x), \ldots, \mu_m(x))$ *attains its local maximum at* $x^* \in X$, *then* x^* *is a global maximizer of* φ *over* X.

Corollary 6.3 *Let* $X \subseteq \mathbf{R}^n$ *be a convex set, let* T *be a strict t-norm, and let* $\mu_i : \mathbf{R}^n \to [0,1]$, $i = 1, 2, \ldots, m$, *be semistrictly* T*-quasiconcave on* X. *If* $\varphi : \mathbf{R}^n \to [0,1]$ *defined by* $\varphi(x) = T_M(\mu_1(x), \ldots, \mu_m(x))$ *attains its local maximum at* $x^* \in X$, *then* x^* *is a global maximizer of* φ *over* X.

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