Church-Rosserova věta

Proposition 2.3.3. $M =_\beta N \;\Leftrightarrow\; \lambda \vdash M = N$.

Proof. ($\Leftarrow$) By induction on the generation of $\vdash$. ($\Rightarrow$) By induction one shows

$$\begin{array}{lcl} M \to_\beta N & \Rightarrow & \lambda \vdash M = N; \\ M \twoheadrightarrow_\beta N & \Rightarrow & \lambda \vdash M = N; \\ M =_\beta N & \Rightarrow & \lambda \vdash M = N. \end{array}$$

∎

Definition 2.3.4.

1. A *β-redex* is a term of the form $(\lambda x.M)N$. In this case $M[x := N]$ is its *contractum*.
2. A λ-term M *is* a *β-normal form* (β-nf) if it does not have a β-redex as subexpression.
3. A term M *has* a β-normal form if $M =_\beta N$ and N is a β-nf, for some N.

Example 2.3.5. $(\lambda x.xx)y$ is not a β-nf, but has as β-nf the term yy.

An immediate property of nf's is the following.

Lemma 2.3.6. *Let $M, M', N, L \in \Lambda$.*

1. *Suppose M is a β-nf. Then*

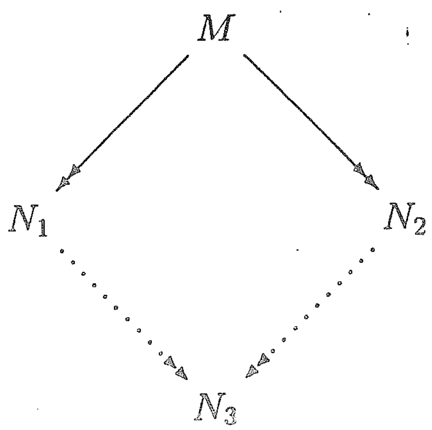
$$M \twoheadrightarrow_\beta N \;\Rightarrow\; N \equiv M.$$

2. *If $M \to_\beta M'$, then $M[x := N] \to_\beta M'[x := N]$.*

Proof. 1. If M is a β-nf, then M does not contain a redex. Hence never $M \to_\beta N$. Therefore if $M \twoheadrightarrow_\beta N$, then this must be because $M \equiv N$.

2. By induction on the generation of $\to_\beta$. ∎

Theorem 2.3.7 (Church–Rosser theorem). *If $M \twoheadrightarrow_\beta N_1, M \twoheadrightarrow_\beta N_2$, then for some N_3 one has $N_1 \twoheadrightarrow_\beta N_3$ and $N_2 \twoheadrightarrow_\beta N_3$; in diagram*



The proof is postponed until 2.3.17.

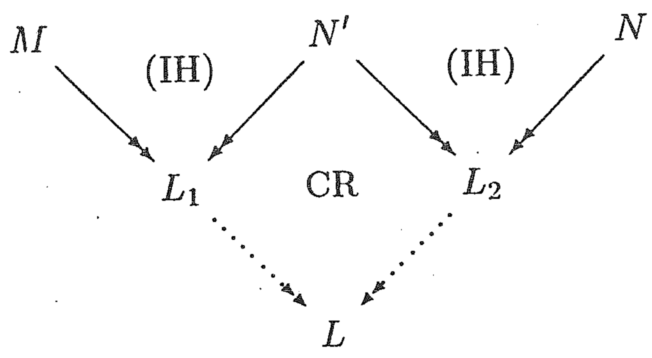
Corollary 2.3.8. *If $M =_\beta N$, then there is an L such that $M \twoheadrightarrow_\beta L$ and $N \twoheadrightarrow_\beta L$.*

Proof. Induction on the generation of $=_\beta$.

Case 1. $M =_\beta N$ because $M \twoheadrightarrow_\beta N$. Take $L \equiv N$.

Case 2. $M =_\beta N$ because $N =_\beta M$. By the IH there is a common β-reduct L_1 of N, M. Take $L \equiv L_1$.

Case 3. $M =_\beta N$ because $M =_\beta N', N' =_\beta N$. Then



■

Corollary 2.3.9.

1. *If M has N as β-nf, then $M \twoheadrightarrow_\beta N$.*
2. *A λ-term has at most one β-nf.*

Proof. 1. Suppose $M =_\beta N$ with N in β-nf. By corollary 2.3.8 one has $M \twoheadrightarrow_\beta L$ and $N \twoheadrightarrow_\beta L$ for some L. But then $N \equiv L$, by Lemma 2.3.6, so $M \twoheadrightarrow_\beta N$.

2. Suppose M has β-nf's N_1, N_2. Then $N_1 =_\beta N_2$ ($=_\beta M$). By Corollary 2.3.8 one has $N_1 \twoheadrightarrow_\beta L, N_2 \twoheadrightarrow_\beta L$ for some L. But then $N_1 \equiv L \equiv N_2$ by Lemma 2.3.6(1). ■

Some consequences.

1. The λ-calculus is consistent, i.e. $\lambda \nvdash$ **true** $=$ **false**. Otherwise **true** $=_\beta$ **false** by Proposition 2.3.3, which is impossible by Corollary 2.3.8

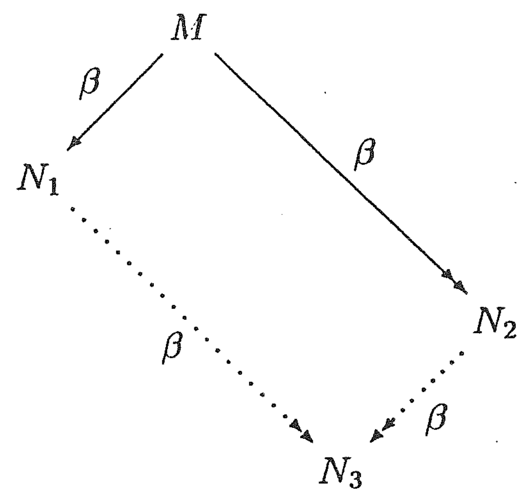
since **true** and **false** are distinct β-nf's. This is a syntactical consistency proof.

2. $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ has no β-nf. Otherwise $\Omega \twoheadrightarrow_\beta N$ with N in β-nf. But Ω only reduces to itself and is not in β-nf.

3. In order to find the β-nf of a term, the various subexpressions of it may be reduced in different orders. If a β-nf is found, then by Corollary 2.3.9 (2) it is unique. Moreover, one cannot go wrong: every reduction of a term can be continued to the β-nf of that term (if it exists). See also Theorem 2.3.20.

Proof of the Church–Rosser theorem

This occupies 2.3.10 - 2.3.17. The idea of the proof is as follows. In order to prove the theorem, it is sufficient to show the following strip lemma:



In order to prove this lemma, let $M \to_\beta N_1$ be a one step reduction resulting from changing a redex R in M in its contractum R' in N_1. If one makes a bookkeeping of what happens with R during the reduction $M \twoheadrightarrow N_2$, then by reducing all 'residuals' of R in N_2 the term N_3 can be found. In order to do the necessary bookkeeping an extended set $\underline{\Lambda} \supseteq \Lambda$ and reduction $\underline{\beta}$ is introduced. The underlining is used in a way similar to 'radioactive tracing isotopes' in experimental biology.

Definition 2.3.10 (Underlining).

1. $\underline{\Lambda}$ is the set of terms defined inductively as follows:

$$\begin{aligned} x \in V &\Rightarrow x \in \underline{\Lambda};\\ M, N \in \underline{\Lambda} &\Rightarrow (MN) \in \underline{\Lambda};\\ M \in \underline{\Lambda}, x \in V &\Rightarrow (\lambda x.M) \in \underline{\Lambda};\\ M, N \in \underline{\Lambda}, x \in V &\Rightarrow ((\underline{\lambda} x.M)N) \in \underline{\Lambda}. \end{aligned}$$

2. Underlined (one step) reduction ($\to_{\underline{\beta}}$ and) $\twoheadrightarrow_{\underline{\beta}}$ are defined starting with the contraction rules

$$(\lambda x.M)N \to M[x := N],$$

$$(\underline{\lambda} x.M)N \to M[x := N].$$

Then $\to$ is extended to the compatible relation $\to_{\underline{\beta}}$ (also w.r.t. $\underline{\lambda}$-abstraction) and $\twoheadrightarrow_{\underline{\beta}}$ is the transitive reflexive closure of $\to_{\underline{\beta}}$.

3. If $M \in \underline{\Lambda}$, then $|M| \in \Lambda$ is obtained from M by leaving out all underlinings. For example, $|(\lambda x.x)((\underline{\lambda} x.x)(\lambda x.x))| \equiv \mathsf{I}(\mathsf{II})$.
4. Substitution for $\underline{\Lambda}$ is defined by adding to the schemes in definition 2.1.5(3) the following:

$$((\underline{\lambda} x.M)N)[y := L] \equiv (\underline{\lambda} x.M[y := L])(N[y := L]).$$

Definition 2.3.11. A map $\varphi : \underline{\Lambda} \to \Lambda$ is defined inductively as follows:

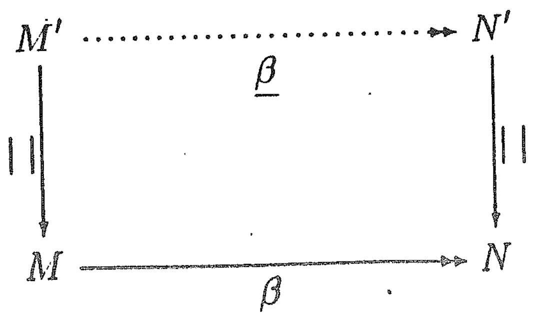
$$\begin{array}{rcl} \varphi(x) & \equiv & x; \\ \varphi(MN) & \equiv & \varphi(M)\varphi(N), \text{ if } M, N \in \underline{\Lambda}; \\ \varphi(\lambda x.M) & \equiv & \lambda x.\varphi(M); \\ \varphi((\underline{\lambda} x.M)N) & \equiv & \varphi(M)[x := \varphi(N)]. \end{array}$$

In other words, the map φ contracts all redexes that are underlined, from the inside to the outside.

Notation 2.3.12. If $|M| \equiv N$ or $\varphi(M) \equiv N$, then this will be denoted by respectively

$$M \xrightarrow[\;||\;]{} N \text{ or } M \xrightarrow[\;\varphi\;]{} N.$$

Lemma 2.3.13.



$M', N' \in \underline{\Lambda}$,
$M, N \in \Lambda$.

Proof. First suppose $M \to_\beta N$. Then N is obtained by contracting a redex in M and N' can be obtained by contracting the corresponding redex in M'. The general statement follows by transitivity. ∎

Lemma 2.3.14. *Let $M, M', N, L \in \underline{\Lambda}$. Then*

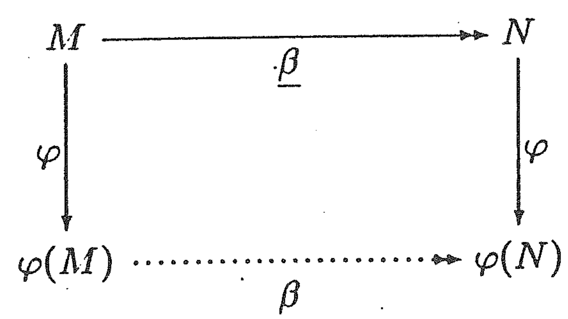
1. *Suppose $x \not\equiv y$ and $x \notin FV(L)$. Then*

$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]].$$

2.

$$\varphi(M[x := N]) \equiv \varphi(M)[x := \varphi(N)].$$

3.



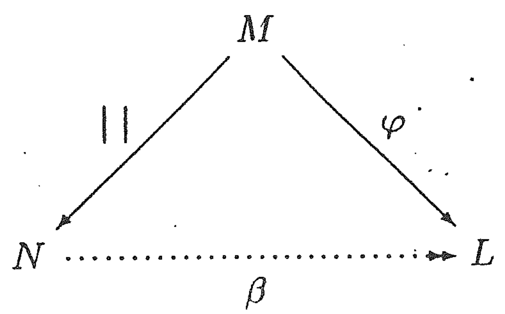
$M, N \in \underline{\Lambda}$.

Proof. 1. By induction on the structure of M.

2. By induction on the structure of M, using (1) in case $M \equiv (\lambda y.P)Q$. The condition of (1) may be assumed to hold by our convention about free variables.

3. By induction on the generation of $\twoheadrightarrow_{\underline{\beta}}$, using (2). ■

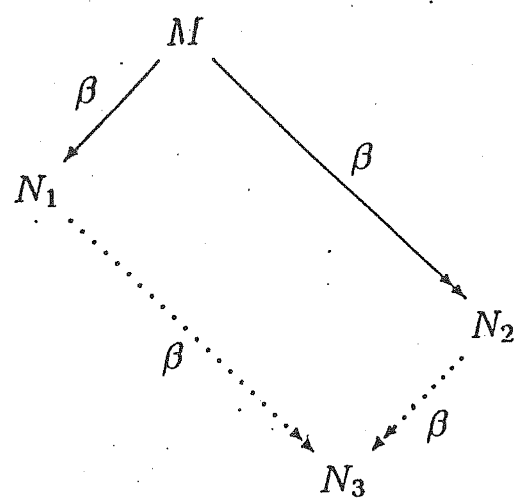
Lemma 2.3.15.



$M \in \underline{\Lambda}$,
$N, L \in \Lambda$.

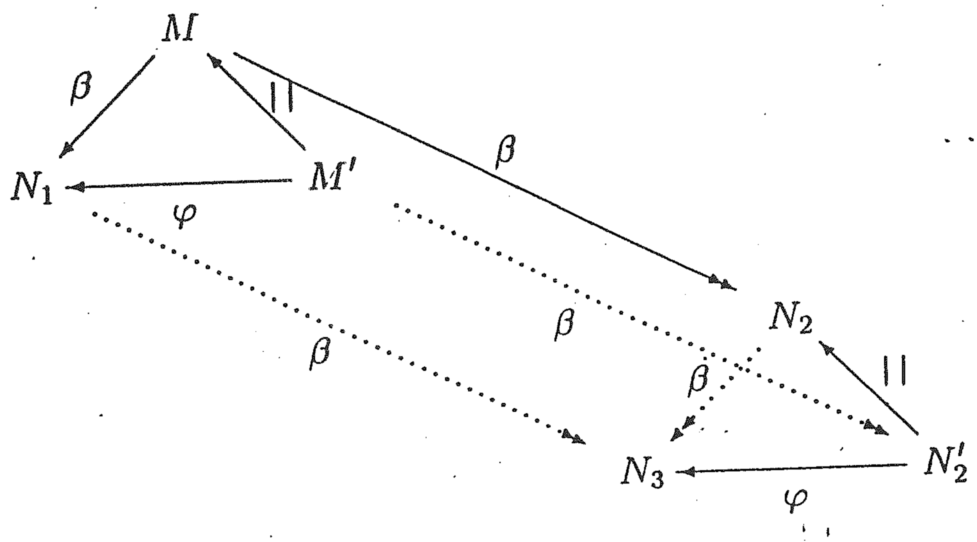
Proof. By induction on the structure of M. ■

Lemma 2.3.16 (Strip lemma).



$M, N_1, N_2, N_3 \in \Lambda.$

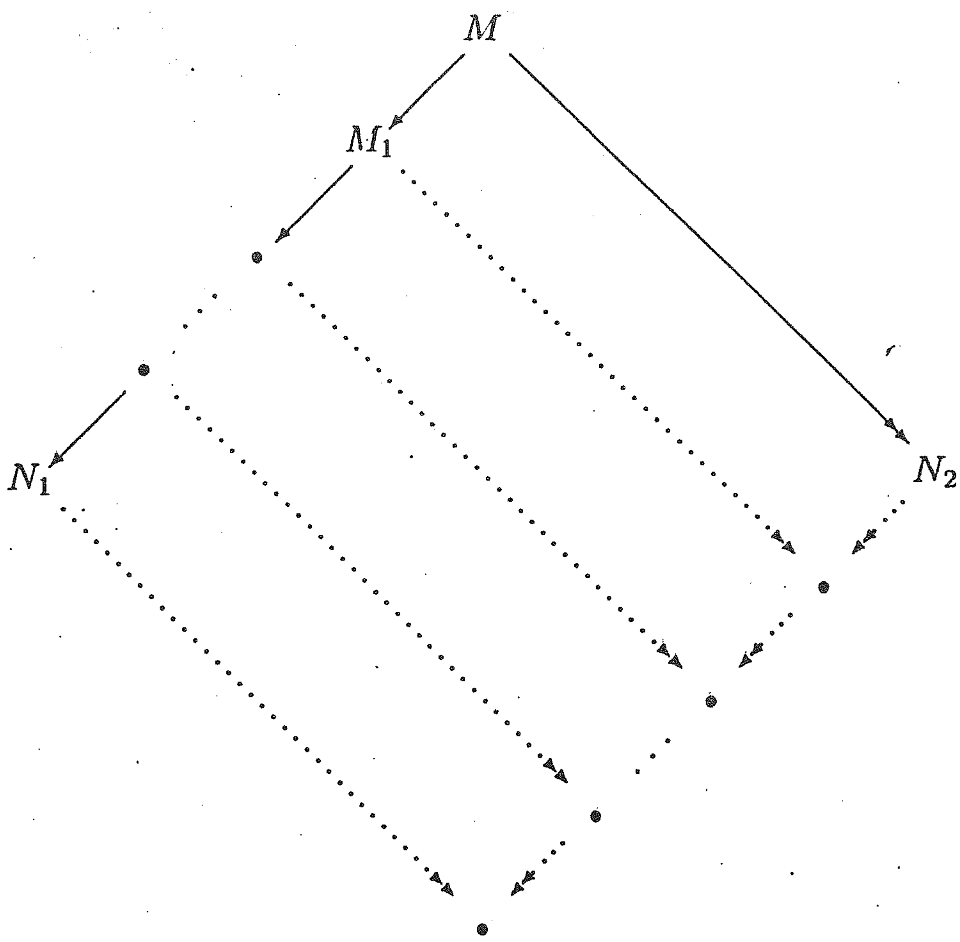
Proof. Let N_1 be the result of contracting the redex occurrence $R \equiv (\lambda x.P)Q$ in M. Let $M' \in \underline{\Lambda}$ be obtained from M by replacing R by $R' \equiv (\underline{\lambda} x.P)Q$. Then $|M'| \equiv M$ and $\varphi(M') \equiv N_1$. By Lemmas 2.3.12, 2.3.13 and 2.3.14 we can construct the following diagram which proves the strip lemma.



■

Theorem 2.3.17 (Church-Rosser theorem). *If $M \twoheadrightarrow_\beta N_1, M \twoheadrightarrow_\beta N_2$, then for some N_3 one has $N_1 \twoheadrightarrow_\beta N_3$ and $N_2 \twoheadrightarrow_\beta N_3$.*

Proof. If $M \twoheadrightarrow_\beta N_1$, then $M \equiv M_0 \to_\beta M_1 \to_\beta \ldots M_n \equiv N_1$. Hence the CR property follows from the strip lemma and a simple diagram chase:

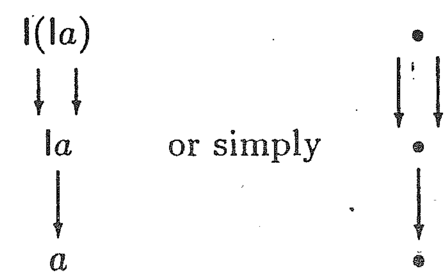


■

Normalization

Definition 2.3.18. For $M \in \Lambda$ the *reduction graph* of M, notation $G_\beta(M)$, is the directed multigraph with vertices $\{N \mid M \twoheadrightarrow_\beta N\}$ and directed by $\rightarrow_\beta$. We have a multigraph because contractions of different redexes are considered as different edges.

Example 2.3.19. $G_\beta(\mathsf{I}(\mathsf{I}a))$ is



A lambda term M is called *strongly normalizing* iff all reduction sequences starting with M terminate (or equivalently iff $G_\beta(M)$ is finite). There are terms that do have an nf, but are not strongly normalizing because they have an infinite reduction graph. Indeed, let $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$.