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LAMBDA CALCULUS

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4 Böhm's Theorem

Böhm's theorem was proved in the late '60s and remains possibly the most significant discovery in the syntax of untyped λ-calculus. It gives rise to a powerful technique for obtaining separability results.

4.1 The theorem and its significance

Theorem 4.1.1 (Böhm) *Let s and t be closed normal λ-terms that are not $\beta\eta$-equivalent. Then there exist closed terms $u_1, \cdots, u_k$ such that*

$$\begin{cases} s\vec{u} &= \mathbf{f} \\ t\vec{u} &= \mathbf{t}. \end{cases}$$

where $\mathbf{t} \equiv \lambda xy.x$ *and* $\mathbf{f} \equiv \lambda xy.y$. □

Exercise 4.1.2 Show that $\mathbf{t}$ and $\mathbf{f}$ of the theorem can be replaced by any pair of closed β-normal forms that are not $\beta\eta$-equivalent.

Böhm's theorem is a classic result in the syntax of untyped λ-calculus. It is a powerful separability result.

An aside on λ-theories

A λ-theory is a consistent extension of $\lambda\beta$ that is closed under provability. A (closed) *equation* is a formula of the form $s = t$ where s and t are closed λ-terms. If $\mathcal{T}$ is a set of closed equations, then the theory $\lambda\beta + \mathcal{T}$ is obtained from $\lambda\beta$ by augmenting the axioms by $\mathcal{T}$.

Definition 4.1.3 Let $\mathcal{T}$ be a set of closed equations. $\mathcal{T}^+$ is the set of closed equations provable in $\lambda\beta + \mathcal{T}$. We say that $\mathcal{T}$ is a ***λ-theory*** just in case $\mathcal{T} = \mathcal{T}^+$ and $\mathcal{T}$ is *consistent* (i.e. there are terms s and t such that $s = t$ is not provable in $\mathcal{T}$).

Corollary 4.1.4 *Any λ-theory which identifies any two closed normal λ-terms that are not $\beta\eta$-equivalent is inconsistent.* □

Proof Take any λ-terms A and B. Write

$$D \equiv \lambda xyz.zyx.$$

Then we have

$$DAB\mathbf{f} = A$$

$$DAB\mathbf{t} = B.$$

Hence if $\mathcal{L} \vdash s = t$ where s and t are any closed normal λ-terms that are not $\beta\eta$-equivalent, then for the $\vec{u}$ given by the theorem, we have $\mathcal{L} \vdash DAB(s\vec{u}) = DAB(t\vec{u})$, and so,

$$\mathcal{L} \vdash A = B.$$

□

The so-called "Böhm-out technique" is crucial to the proof of most ***local structure characterization theorems*** of λ-models.

4.2 Proof of the theorem

First some notations. The ***permutator of order*** n is defined to be the following term

$$\alpha_n \stackrel{\text{def}}{=} \lambda x_1 \cdots x_n x.xx_1 \cdots x_n.$$

Definition 4.2.1 We shall call ***Böhm transformation*** any function from Λ (the collection of λ-terms) to Λ defined by composing basic functions of the form $t \mapsto tu_0$ or $t \mapsto t[u_0/x]$ where u_0 and x are a given term and variable respectively.

We shall denote the functions as follows:

$$\begin{aligned} \mathbf{B}_{u_0} &: t \mapsto tu_0 \\ \mathbf{B}_{u_0,x} &: t \mapsto t[u_0/x]. \end{aligned}$$

Lemma 4.2.2 *For every Böhm transformation B, there are terms $u_1, \cdots, u_k$ such that $Bs = su_1 \cdots u_k$ for every closed term s.* □

Exercise 4.2.3 Prove the lemma.

Lemma 4.2.4 *Let s, t be two λ-terms. If one of the following*

$$\begin{aligned} &(1) \quad s \equiv xs_1 \cdots s_p \\ & \quad t \equiv yt_1 \cdots t_q \qquad \text{where } x \neq y \text{ or } p \neq q \\ &(2) \quad s \equiv \lambda x_1 \cdots x_m x.xs_1 \cdots s_p \\ & \quad t \equiv \lambda x_1 \cdots x_n x.xt_1 \cdots t_q \quad \text{where } m \neq n \text{ or } p \neq q \end{aligned}$$

holds then

$$\begin{cases} Bs = \mathbf{f} \\ Bt = \mathbf{t} \end{cases}$$

for some Böhm transformation B.

Proof **Case (1):**

(i) $x \neq y$, take $\sigma \equiv \lambda z_1 \cdots z_p.\mathbf{f}$ and $\tau \equiv \lambda z_1 \cdots z_q.\mathbf{t}$. Take B to be $\mathbf{B}_{\sigma,x} \circ \mathbf{B}_{\tau,y}$. Then

$$\begin{aligned} Bs &= \mathbf{f} \\ Bt &= \mathbf{t}. \end{aligned}$$

(ii) $x = y$ and $p < q$. Then

$$\begin{aligned} \mathbf{B}_{\alpha_q,x}s &= \alpha_q s_1^* \cdots s_p^* &&= \lambda z_{p+1} \cdots z_q z.zs_1^* \cdots s_p^* z_{p+1} \cdots z_q \\ \mathbf{B}_{\alpha_q,x}t &= \alpha_q t_1^* \cdots t_q^* &&= \lambda z.zt_1^* \cdots t_q^* \end{aligned}$$

where (-)* means (-)$[\alpha_q/x]$. This is case (2)(i).

Case (2):

(i) $m \neq n$, say $m < n$; take distinct variables $z_1, \cdots, z_n, z$ not occurring in s, t. Let

$$B \stackrel{\text{def}}{=} \mathbf{B}_z \circ \mathbf{B}_{z_n} \circ \cdots \circ \mathbf{B}_{z_1}.$$

Then

$$Bs = z_{m+1} s_1^* \cdots s_p^* z_{m+2} \cdots z_n z$$

where $(\text{-})^*$ is $(\text{-})[z_1/x_1, \cdots, z_m/x_m, z_{m+1}/x]$, and

$$Bt = z t_1^\dagger \cdots t_q^\dagger$$

where $(\text{-})^\dagger$ is $(\text{-})[z_1/x_1, \cdots, z_n/x_n, z/x]$. This is just case (1)(i).

(ii) $m = n$ and $p \neq q$; let $B \stackrel{\text{def}}{=} \mathbf{B}_x \circ \mathbf{B}_{x_m} \circ \cdots \circ \mathbf{B}_{x_1}$. We have

$$Bs = x s_1 \cdots s_p$$
$$Bt = x t_1 \cdots t_q$$

This is just case (1)(ii).

Note: cases $(2)(ii) \longrightarrow (1)(ii) \longrightarrow (2)(i) \longrightarrow (1)(i)$. □

Theorem 4.2.5 *Let s and t be non-$\beta\eta$-equivalent normal λ-terms, and $x_1, \cdots, x_k$ any distinct variables. Then for any $n_1, \cdots n_k$, provided they are large enough, there is a Böhm transformation B such that*

$$\begin{cases} B(s[\alpha_{n_1}/x_1, \cdots, \alpha_{n_k}/x_k]) = \mathbf{f} \\ B(t[\alpha_{n_1}/x_1, \cdots, \alpha_{n_k}/x_k]) = \mathbf{t}. \end{cases}$$

Proof The size $\mathsf{size}(s)$ of a term s is defined by recursion as follows:

$$\mathsf{size}(x) \stackrel{\text{def}}{=} 1$$
$$\mathsf{size}(st) \stackrel{\text{def}}{=} \mathsf{size}(s) + \mathsf{size}(t)$$
$$\mathsf{size}(\lambda x.s) \stackrel{\text{def}}{=} \mathsf{size}(s) + 2.$$

We prove by induction on $\mathsf{size}(s) + \mathsf{size}(t)$.

Case analysis:

(1) s and t are both abstractions

(2) only one of s and t is an abstraction

(3) both are not abstractions.

Claim: It suffices to consider the last case.

Proof of Claim Take $y \neq x_1, \cdots, x_k$ with no occurrence in s and t, and let w_s and w_t be the normal form of sy and ty respectively. Now w_s is not $\beta\eta$-equivalent to w_t (why?). Suppose case (1), say $s \equiv \lambda x.u$ and $t \equiv \lambda x'.v$ then $w_s \equiv u[y/x]$, $w_t \equiv v[y/x']$ and

$$\mathsf{size}(w_s) + \mathsf{size}(w_t) \quad = \quad \mathsf{size}(s) + \mathsf{size}(t) - 4.$$

Suppose case (2), say, $s \equiv \lambda x.u$ and t is not an abstraction, then either t is a variable or $v_1 v_2$. Thus $w_s \equiv u[y/x]$ and $w_t \equiv ty$ and

$$\mathsf{size}(w_s) + \mathsf{size}(w_t) \quad = \quad \mathsf{size}(s) + \mathsf{size}(t) - 1.$$

Hence, in both cases, we can apply the induction hypothesis to w_s and w_t. Suppose for any $n_1, \cdots, n_k$ there exists B such that

$$\begin{cases} B(w_s[\alpha_{n_1}/x_1, \cdots, \alpha_{n_k}/x_k]) & = \mathsf{f} \\ B(w_t[\alpha_{n_1}/x_1, \cdots, \alpha_{n_k}/x_k]) & = \mathsf{t}. \end{cases}$$

Take the Böhm transformation $B \circ \mathbf{B}_y$ which works for s and t. □

We shall consider the case where both s and t are not abstractions, say

$$\begin{aligned} s &\equiv x s_1 \cdots s_p \\ t &\equiv y t_1 \cdots t_q \end{aligned}$$

where s_i, t_j are all normal forms.

Fix distinct numbers $n_1, \cdots, n_k$ and variables $x_1, \cdots, x_k$. We write

$$(\text{-})^* \quad \text{for} \quad (\text{-})[\alpha_{n_1}/x_1, \cdots, \alpha_{n_k}/x_k].$$

There are three subcases:

Case (i): $x, y \notin \{x_1, \cdots, x_k\}$.

We have

$$\begin{aligned} s^* &\equiv x s_1^* \cdots s_p^* \\ t^* &\equiv y t_1^* \cdots t_q^*. \end{aligned}$$

If $x \neq y$ or $p \neq q$ then result follows from Lemma 4.2.4. If $x = y$ and $p = q$ take any number $n > p, n_1, \cdots, n_k$. Then take $B = \mathbf{B}_z \circ \mathbf{B}_{z_n} \circ \cdots \circ \mathbf{B}_{z_{p+1}} \circ \mathbf{B}_{\alpha_n, x}$. We have

$$\begin{aligned} Bs^* &= z s_1^\dagger \cdots s_p^\dagger z_{p+1} \cdots z_n \\ Bt^* &= z t_1^\dagger \cdots t_p^\dagger z_{p+1} \cdots z_n \end{aligned}$$

where $(\text{-})^\dagger$ is $(\text{-})[\alpha_{n_1}/x_1, \cdots, \alpha_{n_k}/x_k, \alpha_n/x]$.